# New Developments in Tail-Equivalent Linearization method for Nonlinear Stochastic Dynamics

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# Motivation



- Stochasticity and nonlinearity are essential considerations in assessing the reliability of structural and mechanical systems under extreme loads, e.g.
  - Inelastic response to earthquake ground motion
  - Response to wave loading under material and/or geometric nonlinearities
  - Response to turbulent wind
- Existing methods of nonlinear stochastic dynamic analysis are restricted to special cases, or are not well suited for reliability analysis – hence the need for a new method.

# Outline



- Methods for nonlinear stochastic dynamic analysis
- The equivalent linearization method (ELM)
- Basic elements of TELM:
  - Characterization of linear systems
  - The first-order reliability method (FORM)
  - Discretization of stochastic excitation
- The tail-equivalent linearization method (TELM)
- Characteristics of TELM
- Applications of TELM
- Challenges and limitation of TELM
- Concluding remarks

# Methods for nonlinear stochastic dynamic analysis



- Classical methods
  - Fokker-Planck equation
  - Stochastic averaging
  - Moment/cumulant closure
  - Perturbation
- The equivalent linearization method (ELM)
- Monte Carlo simulation

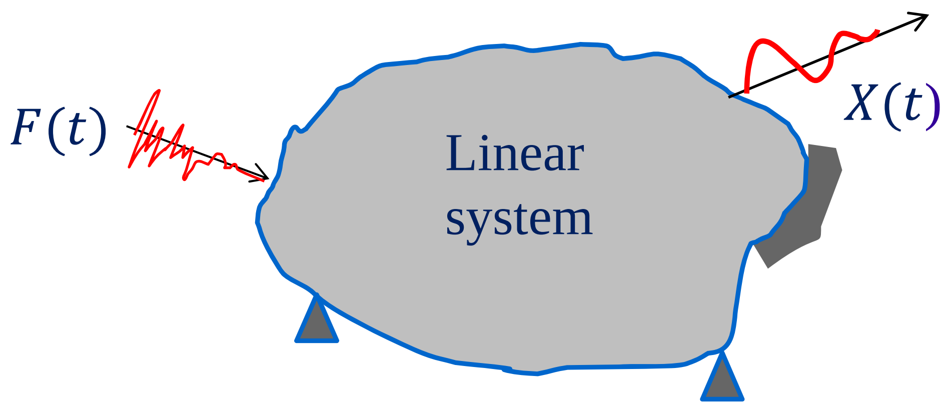
# Equivalent linearization methods (ELM)



- Approximates the nonlinear response in terms of the response of an “equivalent” linear system (Caughey 1963).
- The ELS is determined by minimizing a measure of discrepancy between nonlinear and linear responses:
  - Conventional ELM – minimize the variance of the error between nonlinear and linear responses; requires the assumption of a distribution, typically Gaussian (e.g., Atalik & Utku 1976; Wen 1976).
  - Minimize the difference in mean up-crossing rates at a selected threshold (Casciati et al. 1993).
  - Minimize higher moments of the error (Naess 1995).
  - TELM – set the tail probability of the linear response equal to the first-order approximation of the tail probability of the nonlinear response (Fujimura and Der Kiureghian 2007).

# Characterization of a linear system



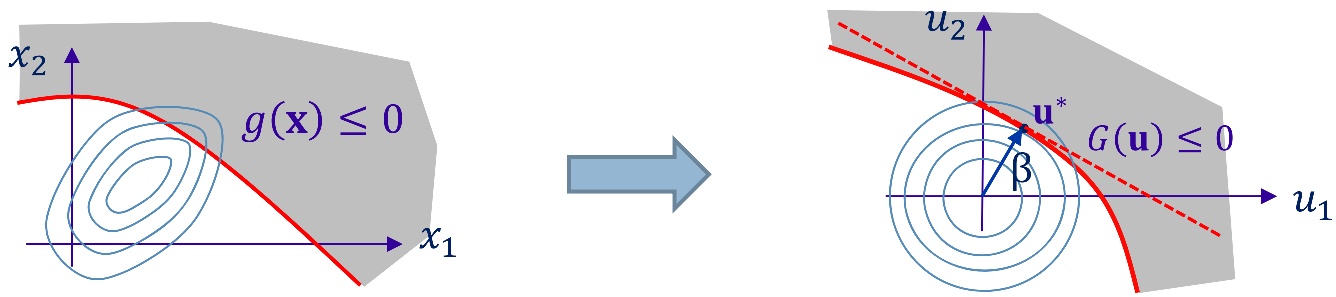



- For an input-output pair $(F(t), X(t))$, a stable linear system is completely defined by either of the following:
  - $h(t) =$ impulse response function (IRF), i.e., response to $F(t) = \delta(t)$
  - $H(\omega) =$ frequency response function (FRF), i.e., amplitude of steady-state response to $F(t) = \exp(i\omega t)$

# The first-order reliability method (FORM)



- An approximate method for solving time-invariant reliability problems:
  - $\mathbf{x}$ = vector of random variables
    $g(\mathbf{x})$ = limit-state function: $\{g(\mathbf{x}) \leq 0\}$ = failure event
    $p_f = \Pr[g(\mathbf{x}) \leq 0]$ probability of failure




  - $\mathbf{u} = \mathbf{u}(\mathbf{x})$ transformation to standard normal space
    $G(\mathbf{u}) = g\big(\mathbf{x}(\mathbf{u})\big)$ limit-state function in transformed space
    $\mathbf{u}^* = \min\arg\{\|\mathbf{u}\| \mid G(\mathbf{u}) = 0\}$ design point
    $\beta = \|\mathbf{u}^*\|$ reliability index
    $p_f \approx \Phi(-\beta)$, FORM approximation

# Discrete representation of a stochastic process



- General form for a zero-mean Gaussian process

  $F(t) = \mathbf{s}(t) \cdot \mathbf{u}$

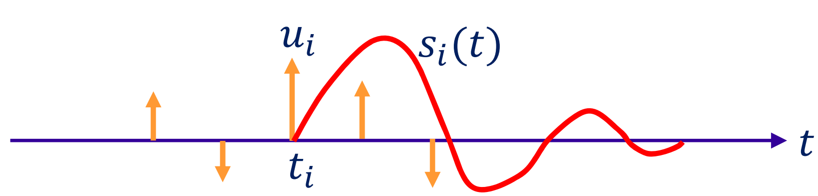
  $\mathbf{s}(t) = [s_1(t) \cdots s_n(t)]$ vector of deterministic basis functions

  $\mathbf{u} = [u_1 \cdots u_n]$ vector of standard normal random variables

  - Time-domain discretization (modulated filtered white noise)

    $s_i(t) = q(t)h_f(t - t_i)$

    $h_f(\cdot) =$ impulse response function of a linear filter



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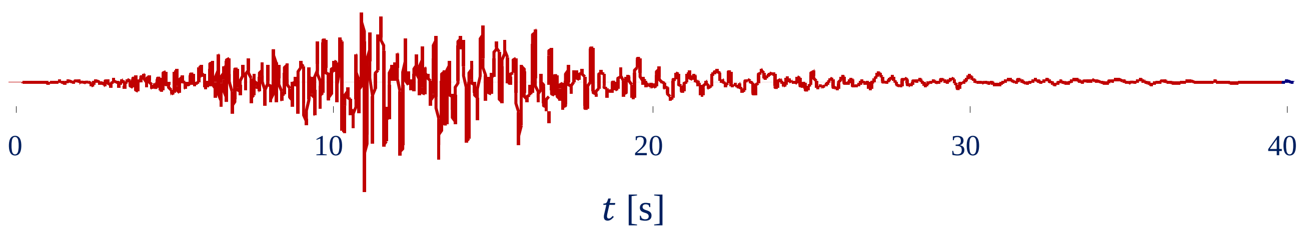

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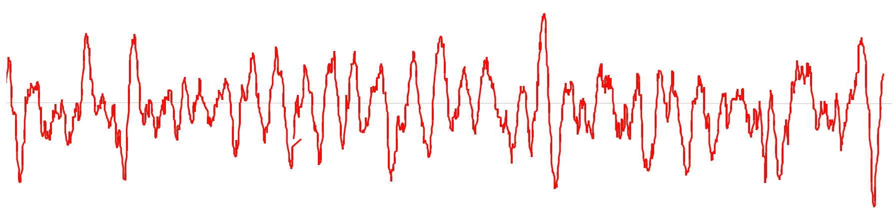
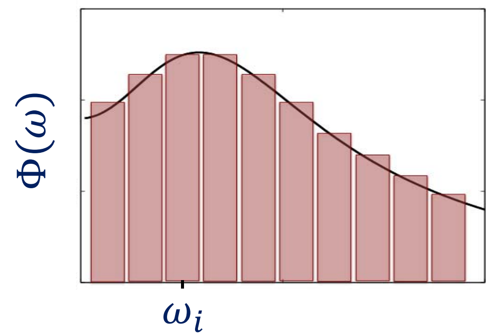
  - Frequency-domain discretization (stationary process)

    $F(t) = \sum_{i=1}^{n/2} \sigma_i [u_i \sin(\omega_i t) + \bar{u}_i \cos(\omega_i t)]$

    $s_i(t) = \sigma_i \sin(\omega_i t),\ \bar{s}_i(t) = \sigma_i \cos(\omega_i t)$



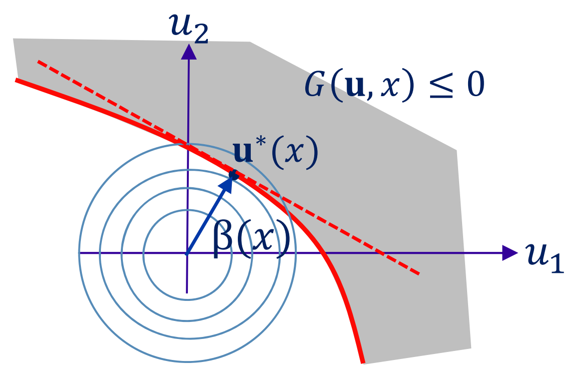


# FORM solution of stochastic dynamic problems



- Definitions:
  - $F(t) = \mathbf{s}(t) \cdot \mathbf{u}$ discretized stochastic excitation
  - $X(t, \mathbf{u})$ = response to discretized stochastic excitation
  - $\Pr\big(x < X(t, \mathbf{u})\big)$ = tail probability for threshold $x$ at time $t$

- Reliability formulation:
  - $G(\mathbf{u}, x) = x - X(t, \mathbf{u})$
  - $\Pr\big(x < X(t, \mathbf{u})\big) = \Pr(G(\mathbf{u}, x) \le 0)$
  - $\mathbf{u}^*(x) = \arg\min\{\|\mathbf{u}\| \mid G(\mathbf{u}, x) = 0\}$
  - $\beta(x) = \|\mathbf{u}^*(x)\|$
  - $\Pr\big(x < X(t, \mathbf{u})\big) \approx \Phi\big(-\beta(x)\big)$ FORM approx. of tail probability



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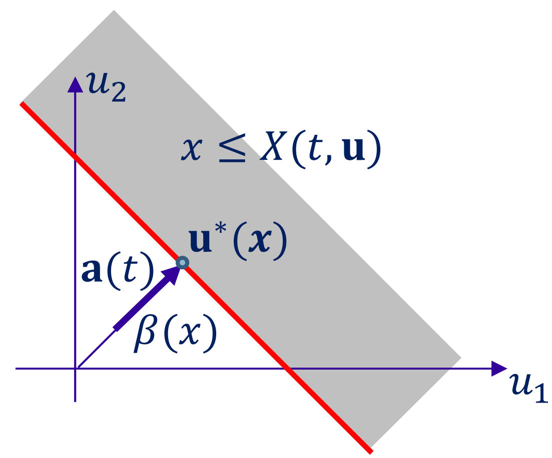
  - $X(t, \mathbf{u}) =$ response to discretized stochastic excitation
  - $\Pr\big(x < X(t, \mathbf{u})\big) =$ tail probability for threshold $x$

- Reliability formulation – the case of linear system
  - $X(t, \mathbf{u}) = \mathbf{a}(t) \cdot \mathbf{u}, \quad a_i(t) =$ response to $s_i(t)$
  - $G(\mathbf{u}, x) = x - \mathbf{a}(t) \cdot \mathbf{u}$
  - $\mathbf{u}^*(x) = \dfrac{x\mathbf{a}(t)}{\|\mathbf{a}(t)\|^2}$
  - $\beta(x) = \dfrac{x}{\|\mathbf{a}(t)\|}$
  - $\Pr\big(x < X(t, \mathbf{u})\big) = \Phi\big(-\beta(x)\big)$

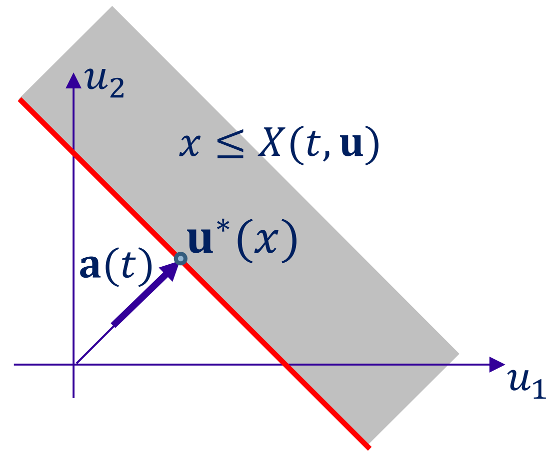


# Identification of the linear system



- Given the design point, one can identify the linear system (for the particular input-output pair)

$$\mathbf{u}^* \quad \rightarrow \quad \mathbf{a}(t) = \frac{x\mathbf{u}^*}{\|\mathbf{u}^*\|^2}$$




  - Time-domain analysis:

    Solve for $h(t)$ in system of equations

    $$\sum_{j=1}^{n} h(t - t_j) s_i(t_j) \Delta t = a_i(t), \;\; i = 1, \cdots, n$$

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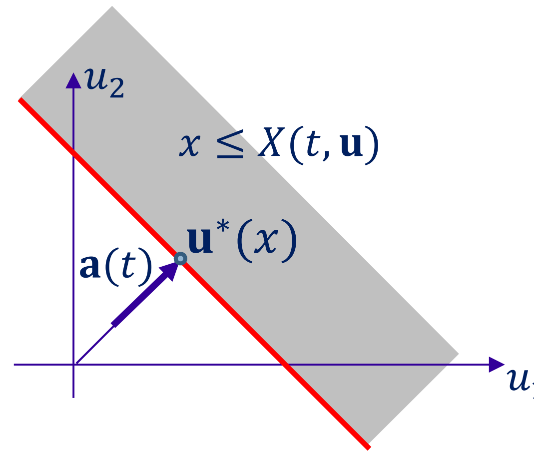



  - Frequency-domain analysis:

$$|H(\omega_i)| = \frac{\sqrt{a_i(t)^2 + \bar{a}_i(t)^2}}{\sigma_i}$$

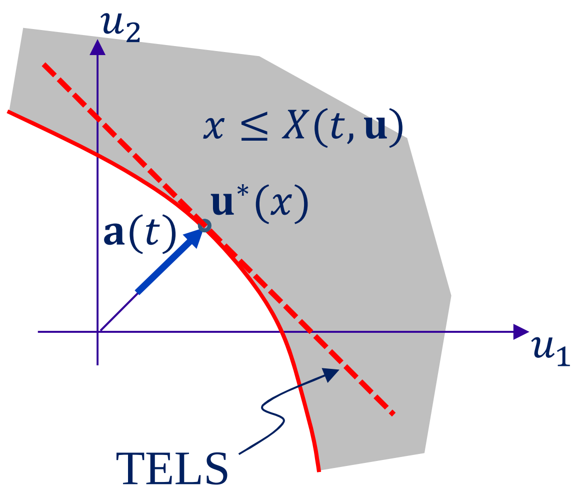
$$\theta_i = \tan^1 \left[\frac{a_i(t)}{a_i(t)}\right]$$

$$H(\omega_i) = |H(\omega_i)| \exp(\mathrm{i}\theta_i)$$

# The tail-equivalent linearization method (TELM)



- For selected threshold $x$ and time $t$, formulate the tail probability problem in terms of the limit-state function $G(\mathbf{u},t) = x - X(t,\mathbf{u})$
- Find the design point $\mathbf{u}^*(x)$
- Find the gradient vector of the tangent plane $\mathbf{a}(t) = \dfrac{x\mathbf{u}^*}{\|\mathbf{u}^*\|^2}$
- Identify the tail-equivalent linear system (TELS) in terms of its IRF $h(t)$ or its FRF $H(\omega)$



# Characteristics of TELM



- For the given threshold $x$ and time $t$,
  the tail probability of the TELS response
  $=$ first-order approximation of the tail probability
  of the nonlinear system response

  ⇒ *Tail-Equivalent Linearization Method* (TELM)

# Characteristics of TELM



- As opposed to ELM and other linearization methods, TELM is a *non-parametric* method.

  ⇒ There is no need to define a parameterized linear system.

  The TELS is identified numerically in terms of its IRF $h(t)$ or its FRF $H(\omega)$.

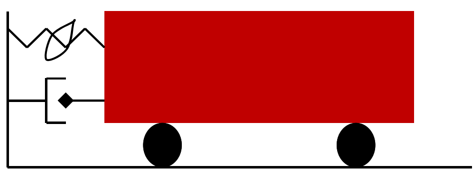
# Design-point excitation and response



- The deign-point excitation $F^*(t) = \mathbf{s}(t) \cdot \mathbf{u}^*$ represents the *most likely realization* of the stochastic excitation to give rise to the event $\{x \leq X(t, \mathbf{u})\}$

- Example:

$$m\ddot{X} + \mathrm{c}\dot{X} + k[\alpha X + (1-\alpha)Z] = F(t)$$
$$\dot{Z} = -\gamma|\dot{X}||Z|^{n-1}Z - \eta|Z|^n\dot{X} + A\dot{X}$$



$m = 3E5\mathrm{kg};\ c = 251\mathrm{kNs/m}$

$k = 2.1E4\ \mathrm{kN/m},\ \alpha = 0.1$

$\gamma = \eta = 1/2\sigma_0,\ n = 3,\ A = 1$

$F(t)$ white noise, $S = 1\mathrm{m}^2/\mathrm{s}^3\mathrm{rad}$

$\sigma_0 = \pi S m^2/ck$

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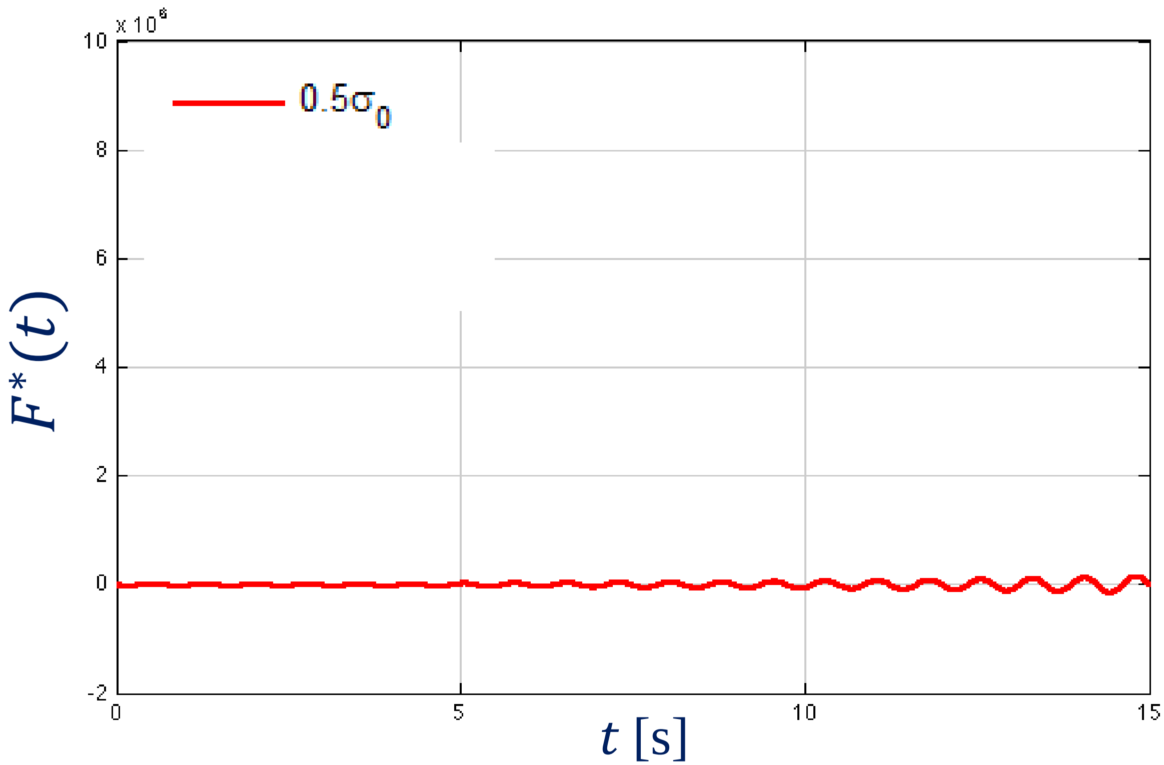

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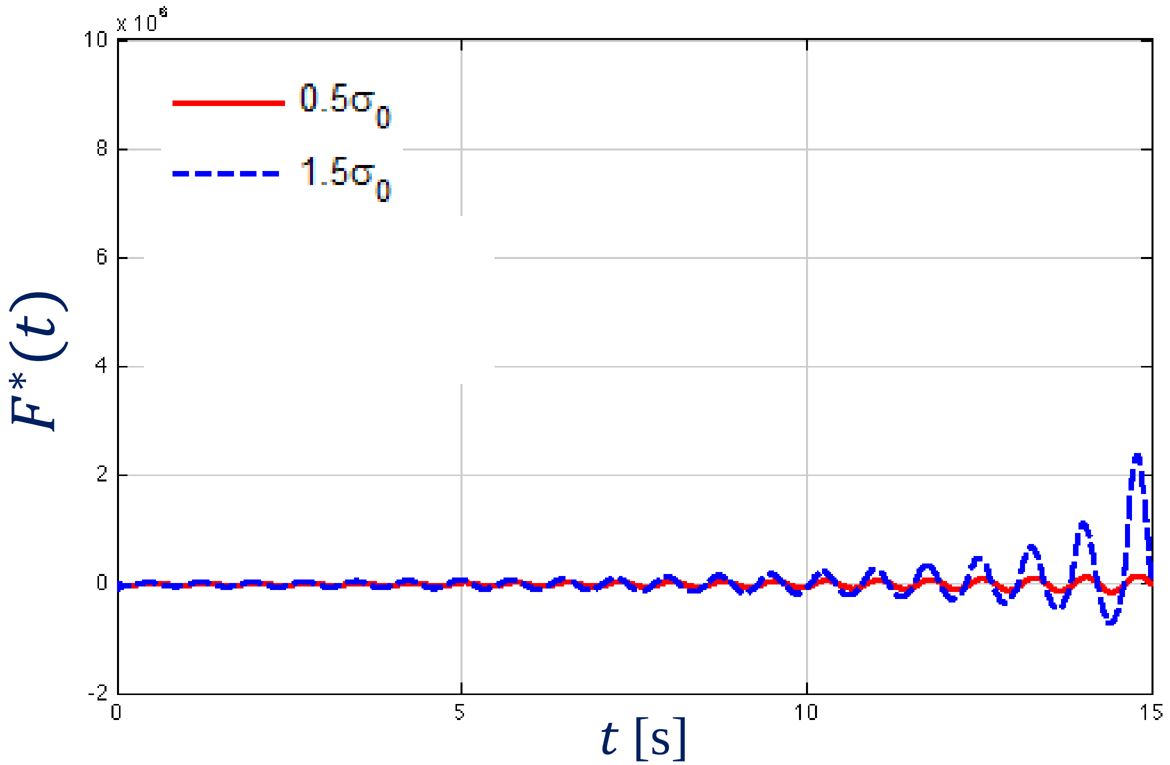

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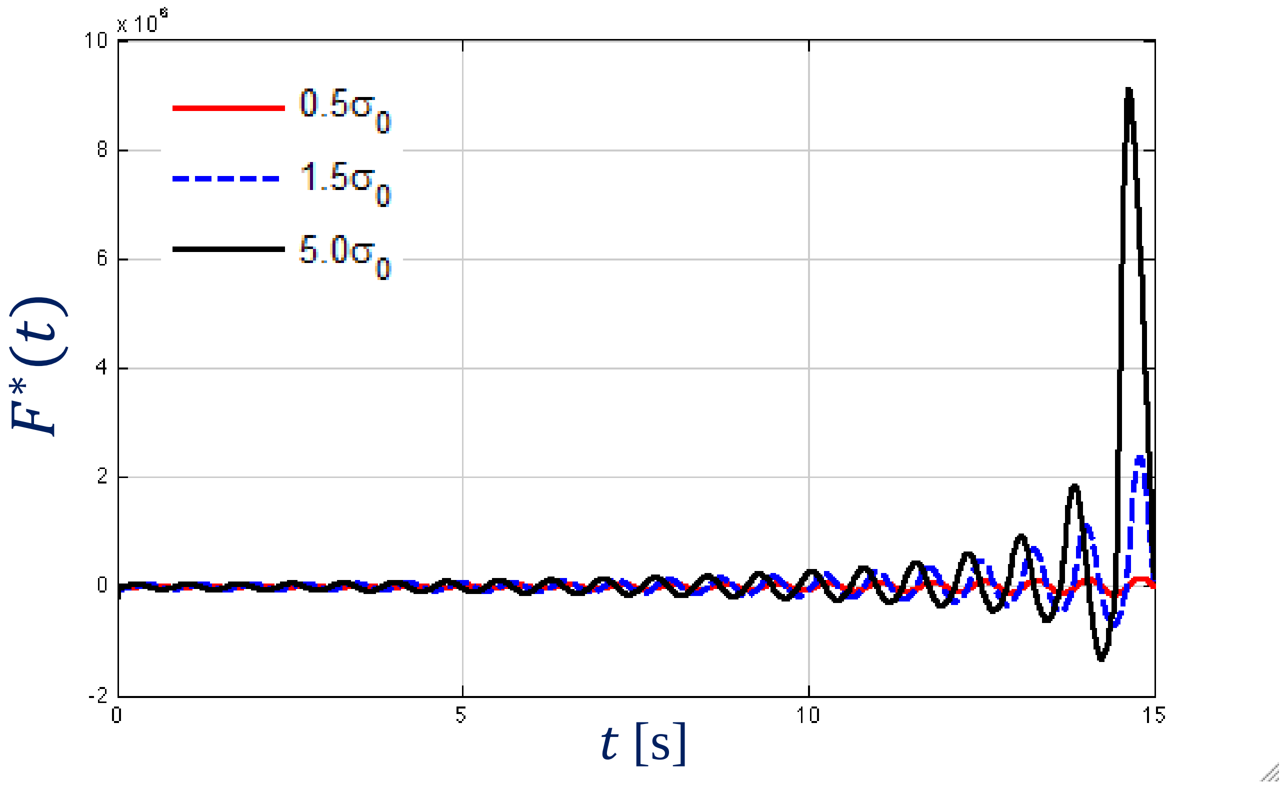
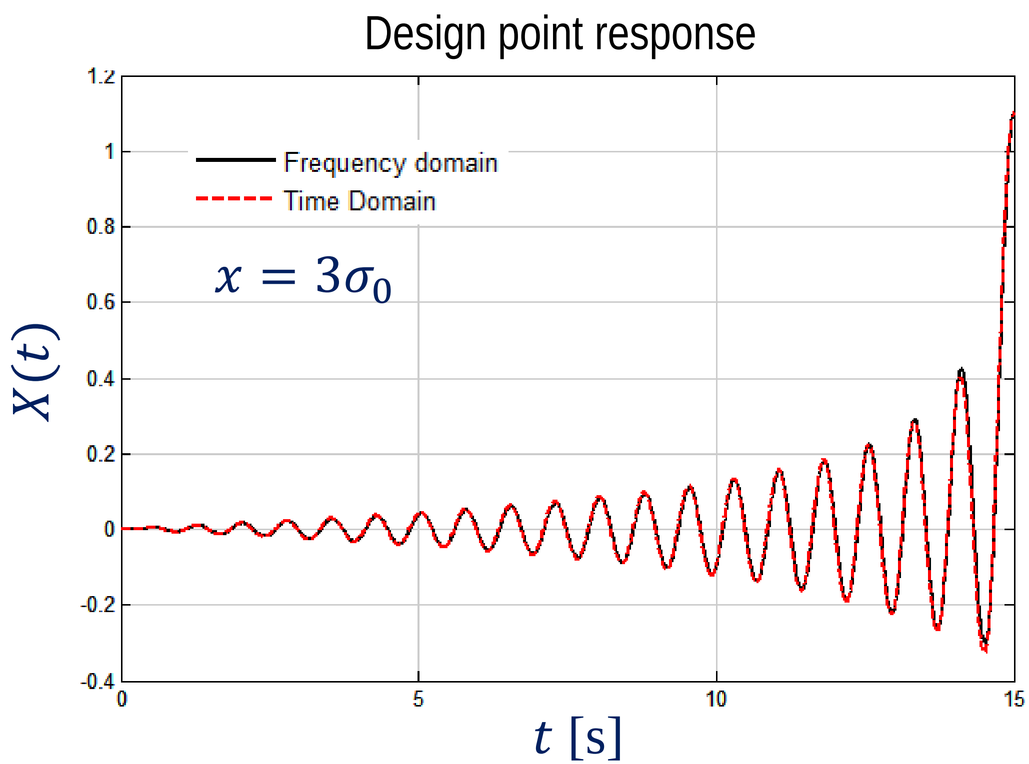
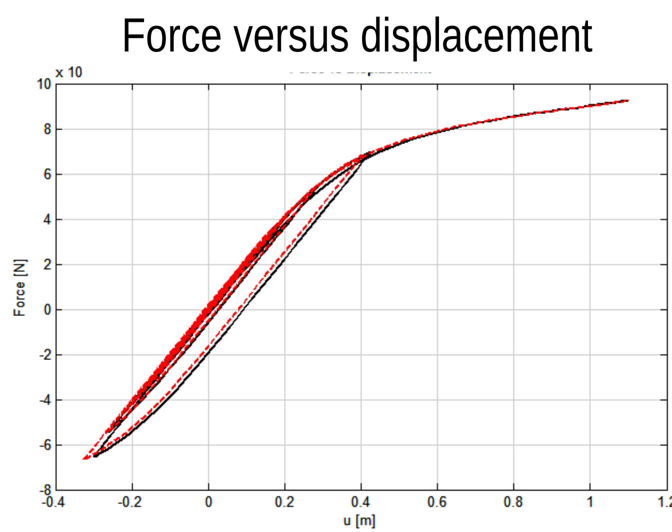


# Design-point excitation and response



- The deign-point response $X(t, \mathbf{u}^*)$ represents the *most likely realization* of the response trajectory leading to $X(t, \mathbf{u}) = x$.

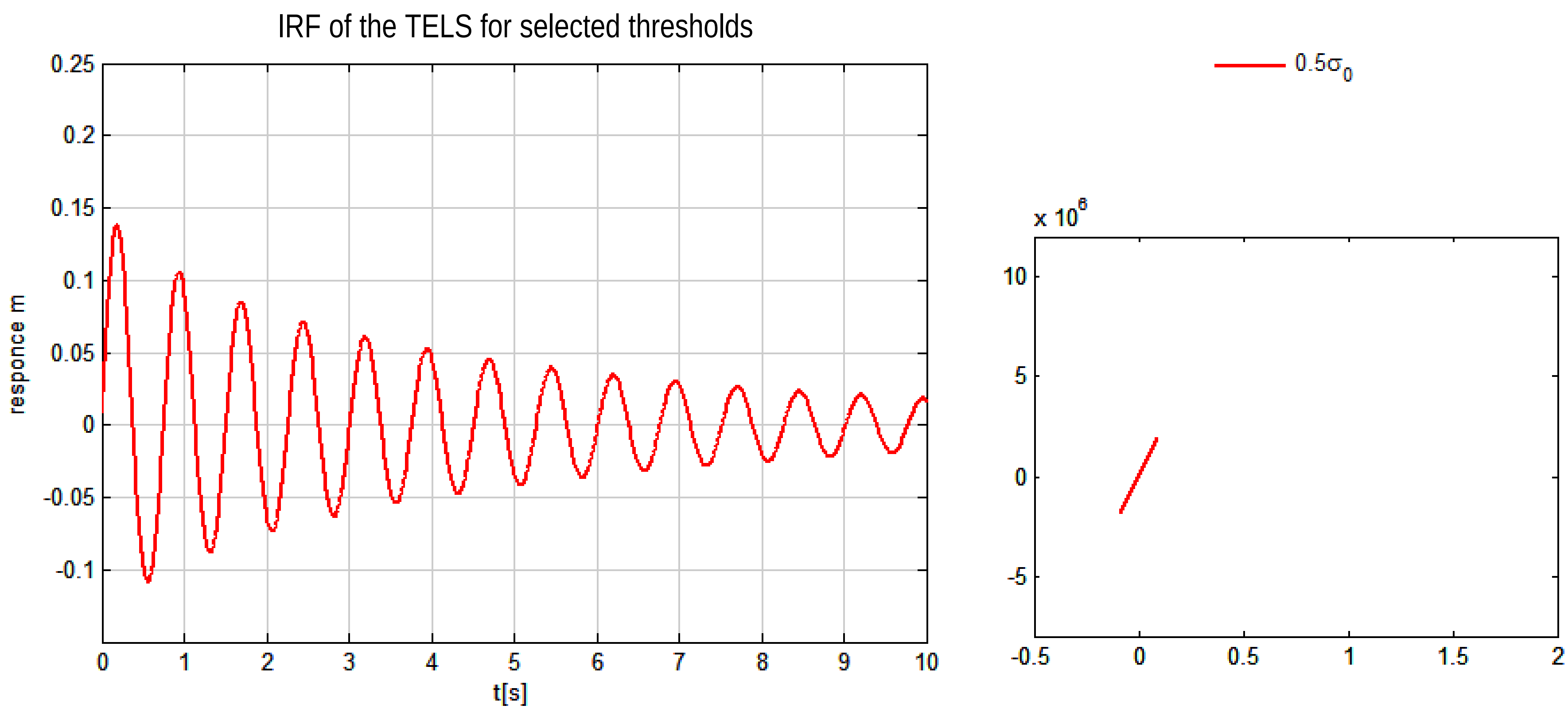


# Characteristics of TELS



- The TELS strongly depends on the selected threshold:
  $h(t) \rightarrow h(t,x), \quad H(\omega) \rightarrow H(\omega,x)$



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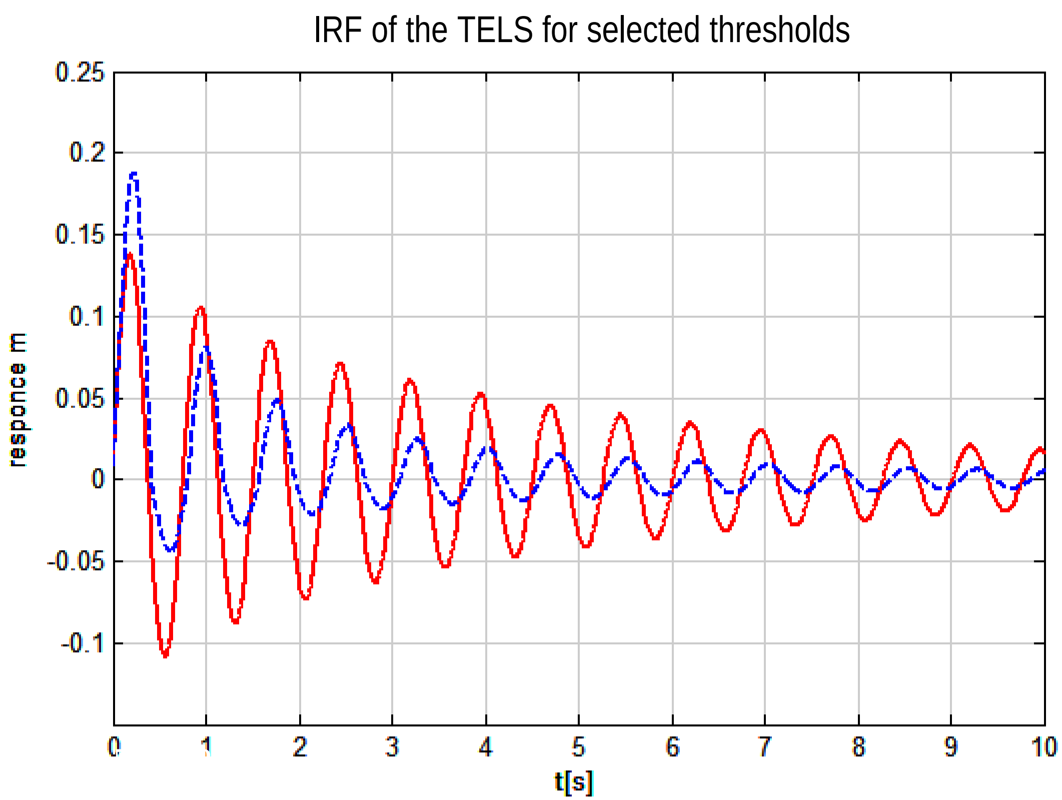


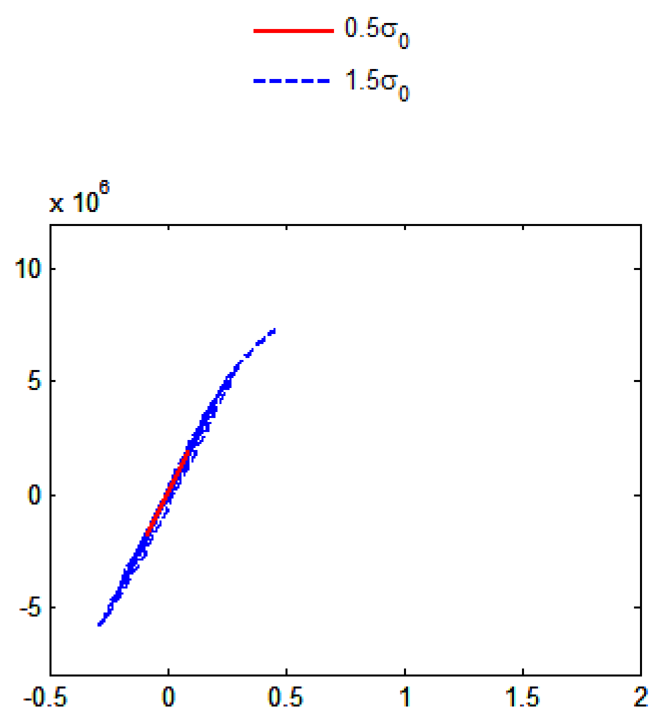

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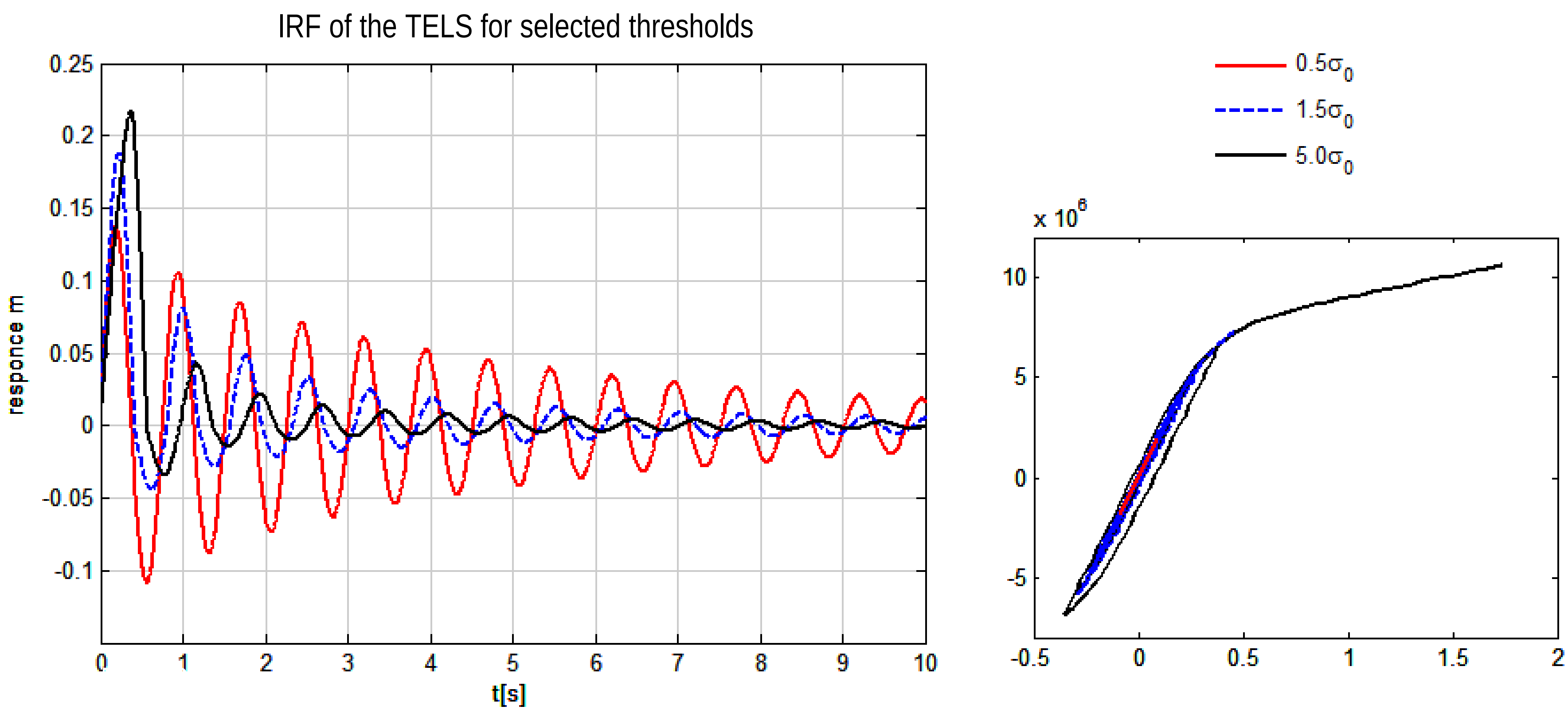

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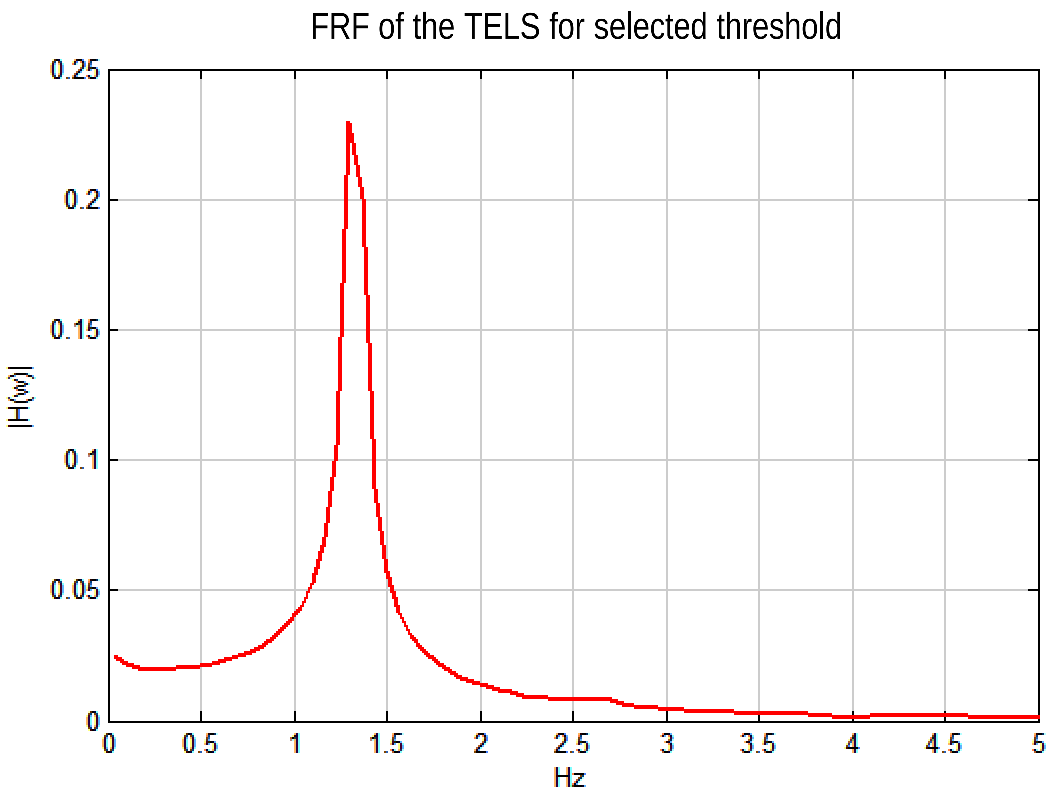


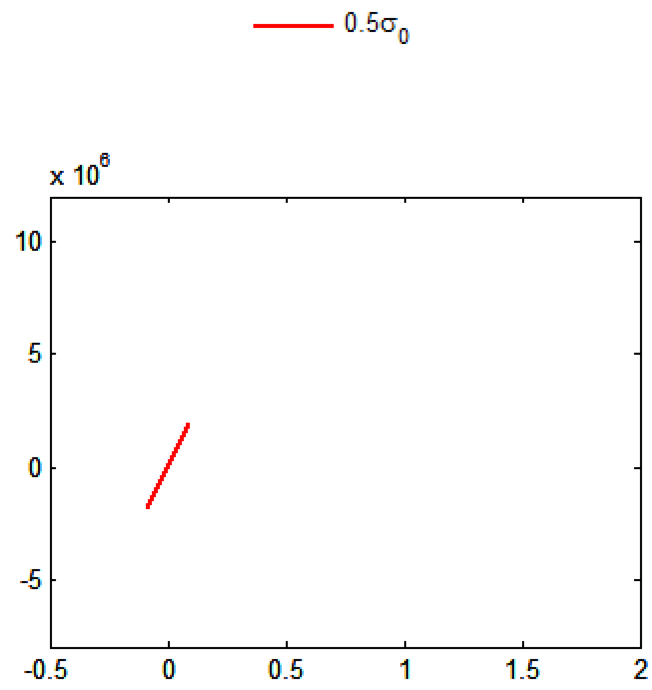

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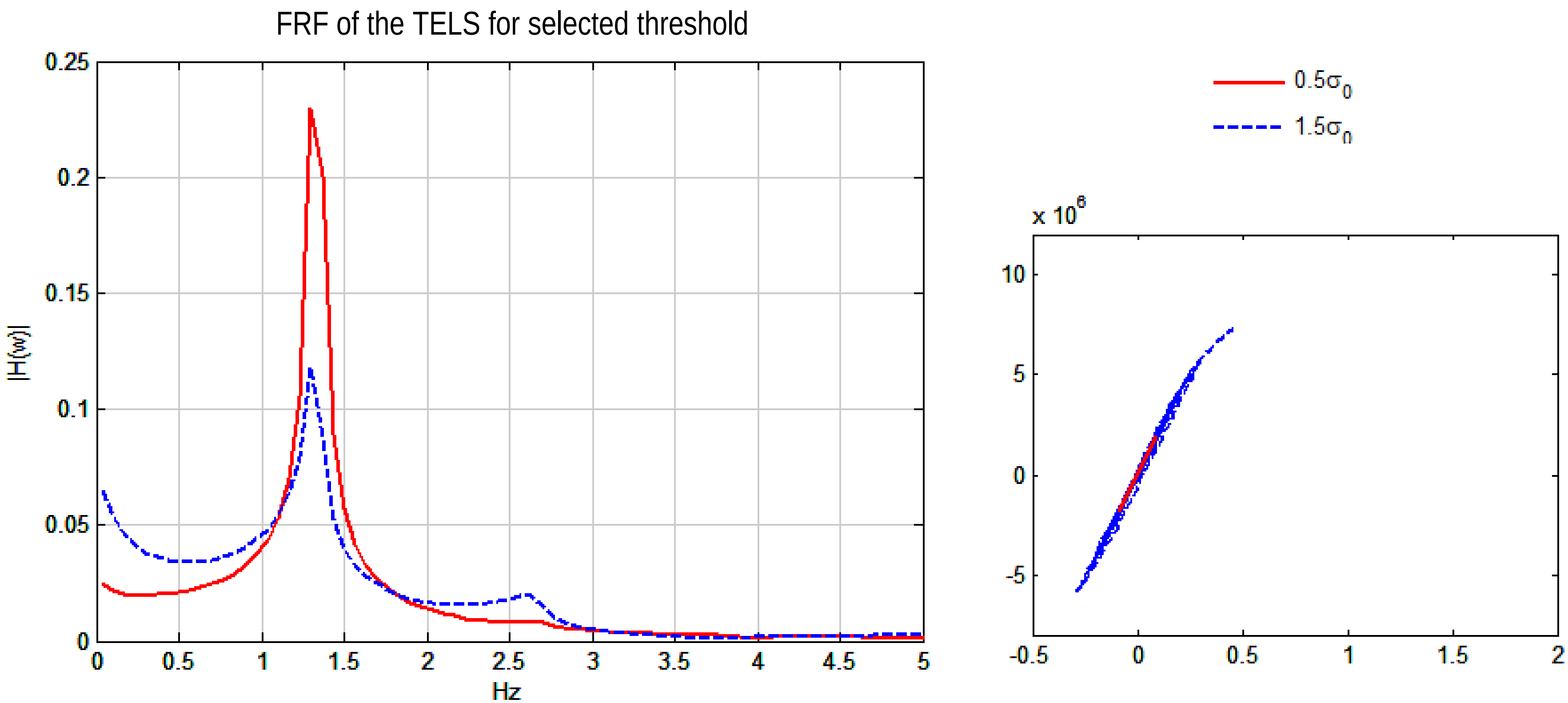

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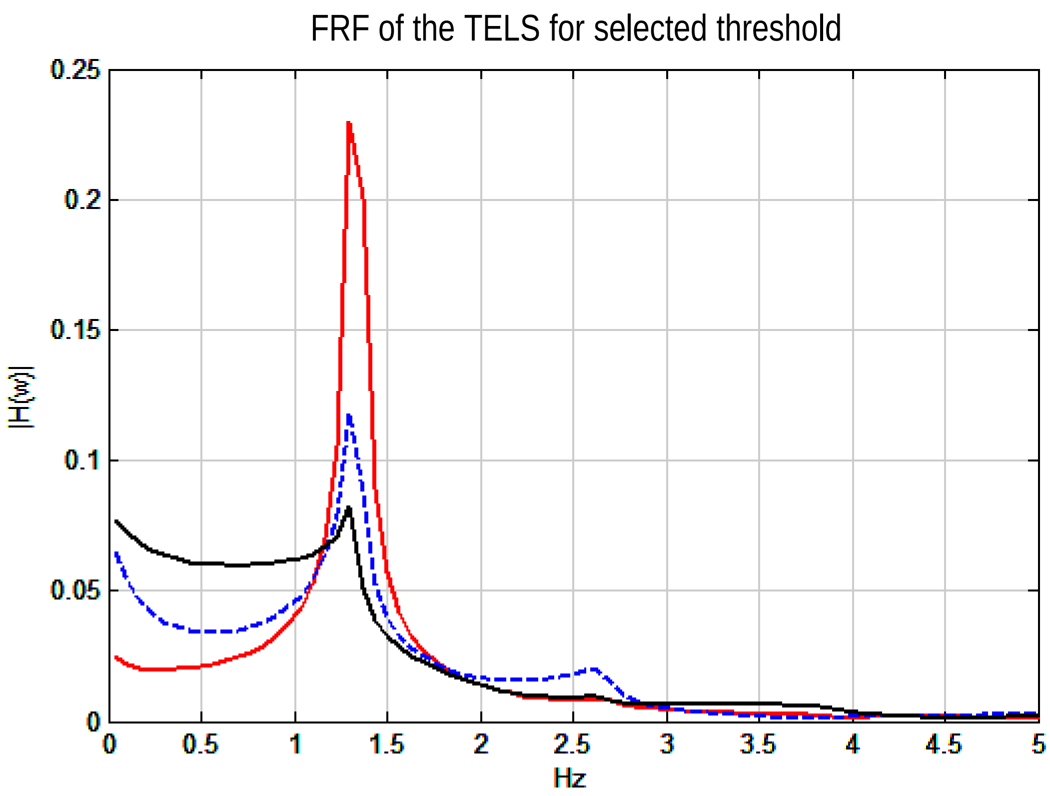
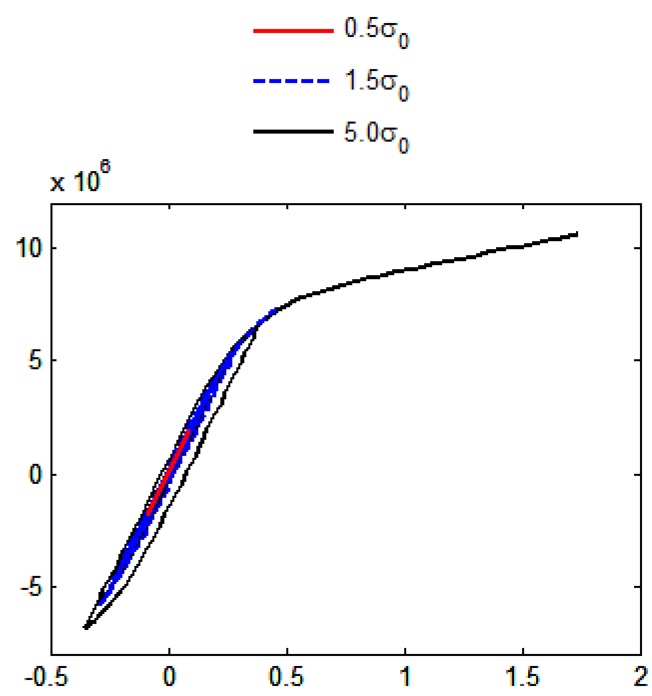


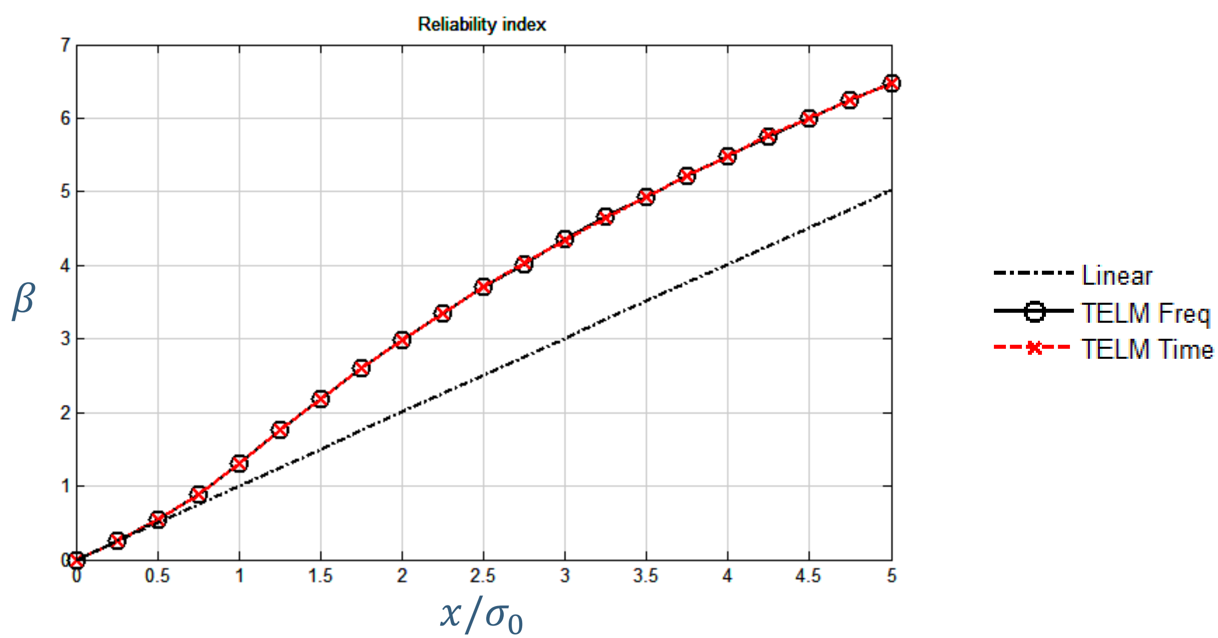



# Characteristics of TELS



- Because of TELS's dependence on the threshold, TELM captures the *non-Gaussian distribution* of the nonlinear response.

$$\Pr[x \leq X(t,u)] \approx \Phi[-\beta(x)]$$



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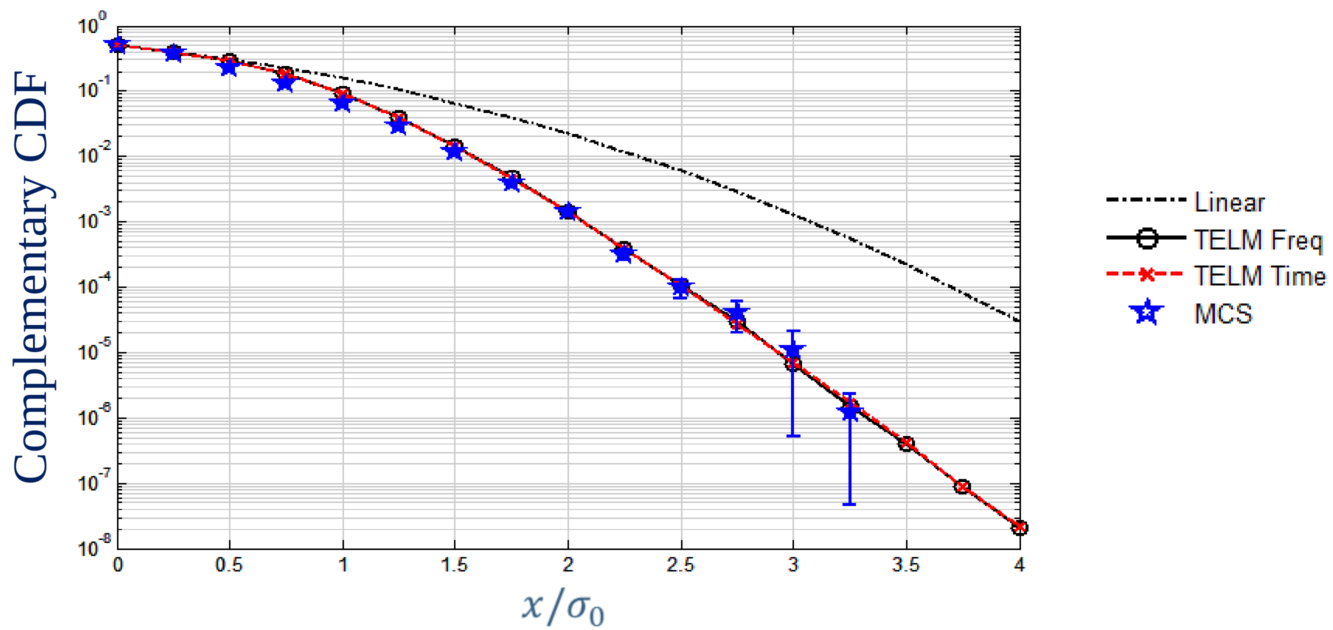

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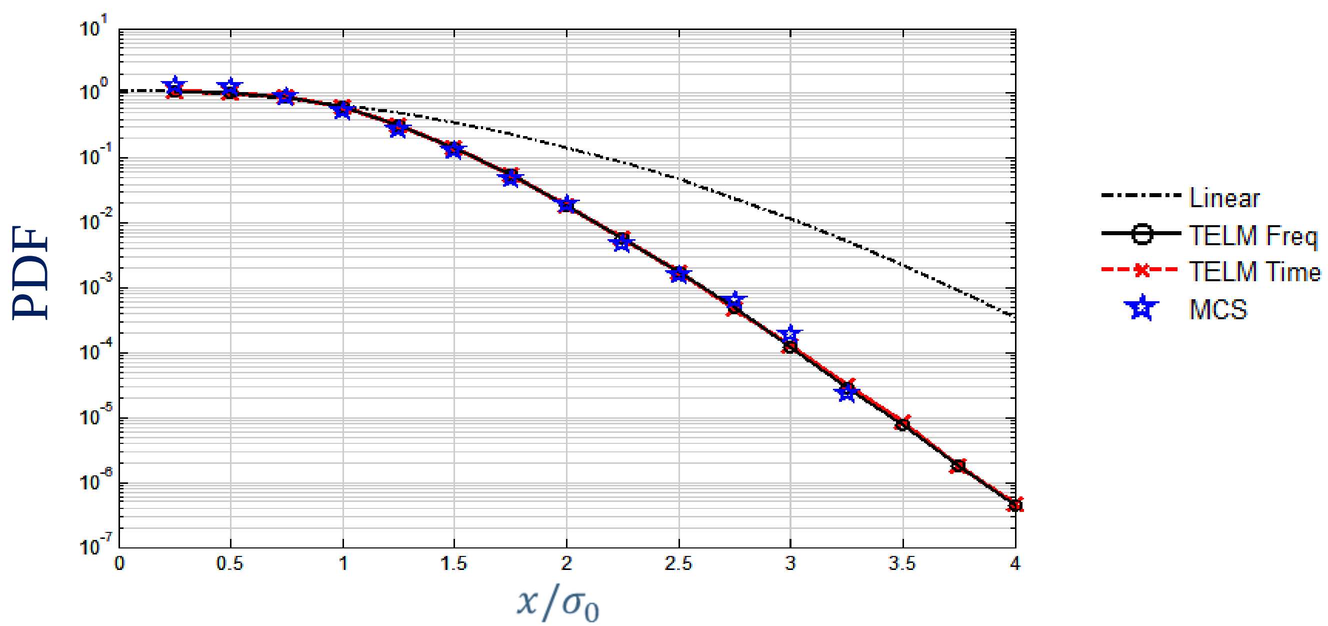
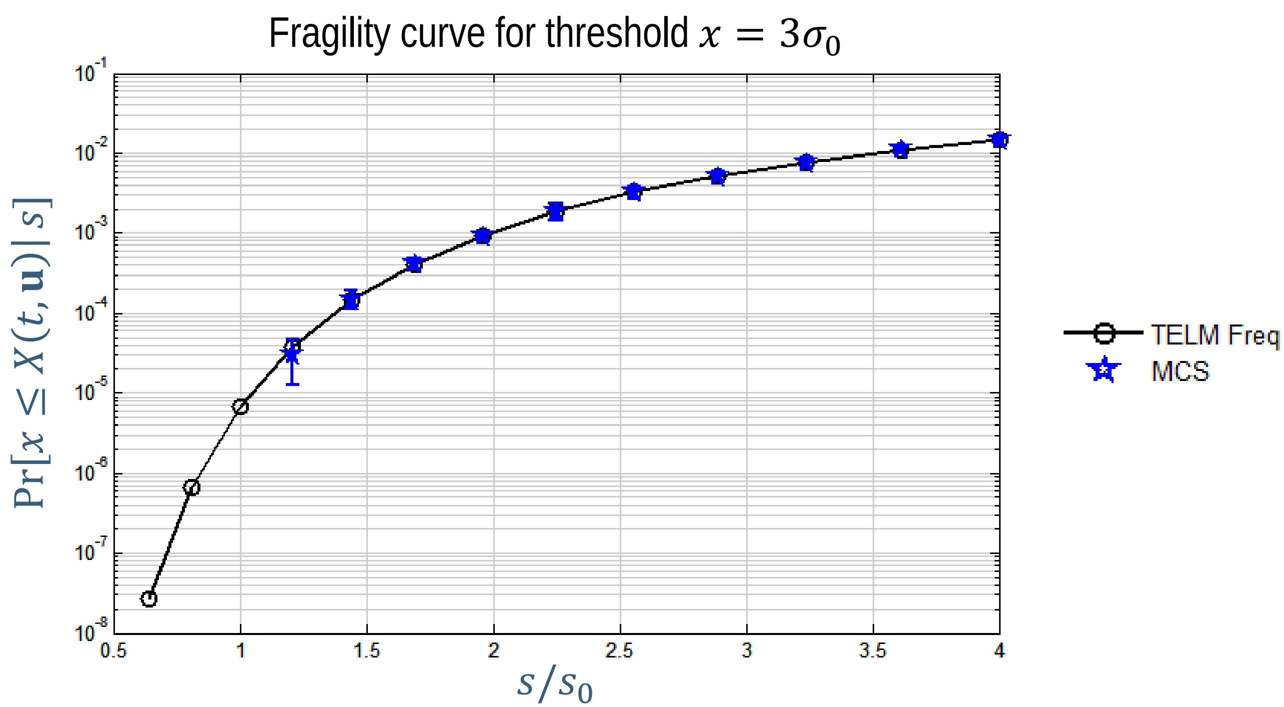


# Characteristics of TELS



- TELS is invariant of the scaling of the excitation, i.e., $h(t, x)$ and $H(\omega, x)$ for excitation $sF(t)$ are invariant of $s$.

Useful property for developing *fragility curves*:

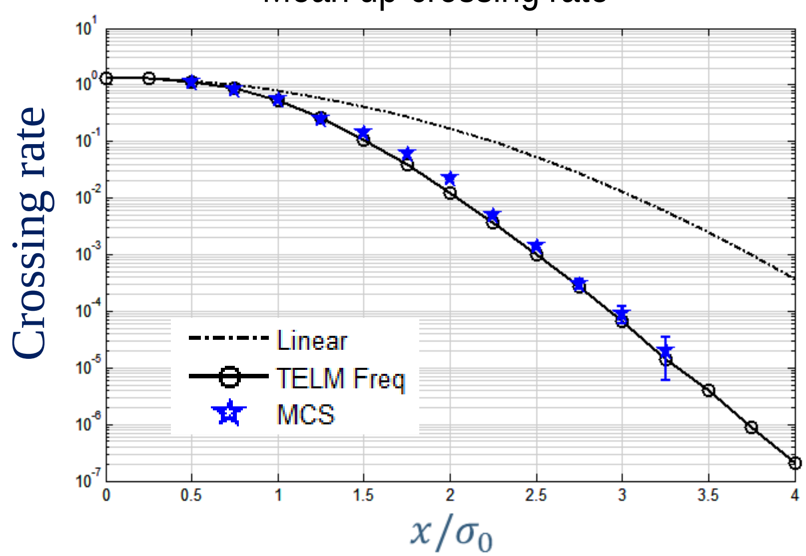


# Characteristics of TELS

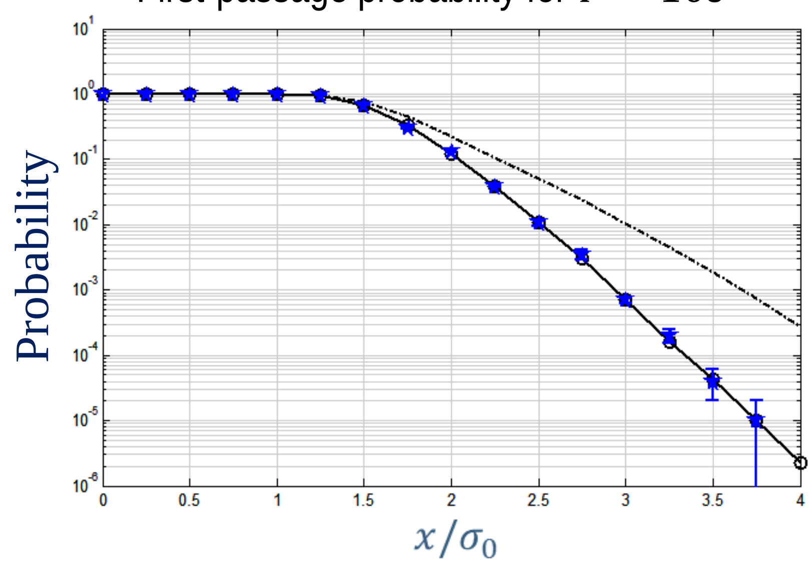


- For stationary response, TELS is invariant of time $t$. Thus, TELSs determined for one time point are sufficient to evaluate all statistical properties of the response, e.g.,
  - *Point-in-time distribution* $\Pr[x \leq X(t, \mathbf{u})]$
  - *Mean up-crossing rate* $\nu^+(x)$
  - *First-passage probability* $\Pr[x \leq \max_{0 \leq t \leq T} |X(t)|]$

Mean up-crossing rate




First-passage probability for $T = 10$s



# Characteristics of TELS



- For non-stationary excitation with smooth evolution in time, TELS at a critical time can be used as an approximation for all times.

  *Evolutionary TELM* (Broccardo/Der Kiureghian 2013)

- Evolutionary input-output for a linear system

  $$\Phi_{XX}(\omega, t) = |M(\omega, t)|^2 \Phi_{FF}(\omega)$$

  $$M(\omega, t) = \int_0^t A(\omega, t-\tau) h(\tau) e^{-\mathrm{i}\omega t} d\tau$$

- ETELM

  $$M(\omega, t, x) \approx \int_0^t A(\omega, t-\tau) \boxed{h_{TELS}(\tau, x)} e^{-\mathrm{i}\omega t} d\tau$$

  $\boxed{h_{TELS}(t, x)}$ determined at a critical point in time, e.g., peak intensity of the excitation.

# Characteristics of TELS



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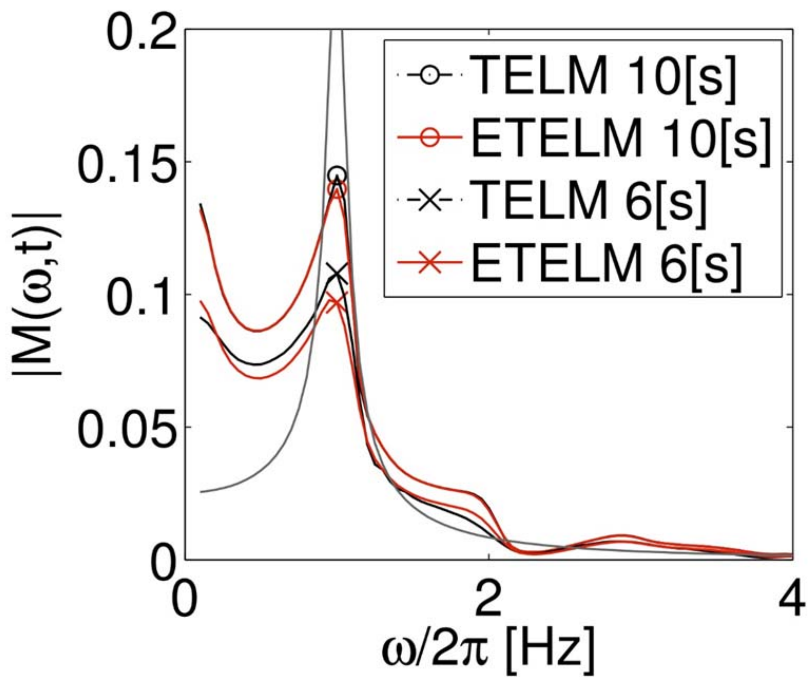
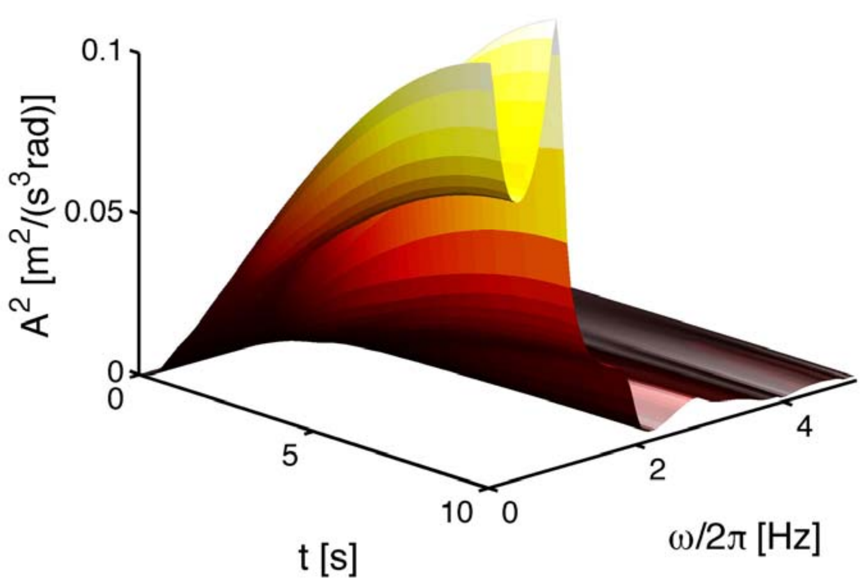
- Example – response to uniformly modulated process

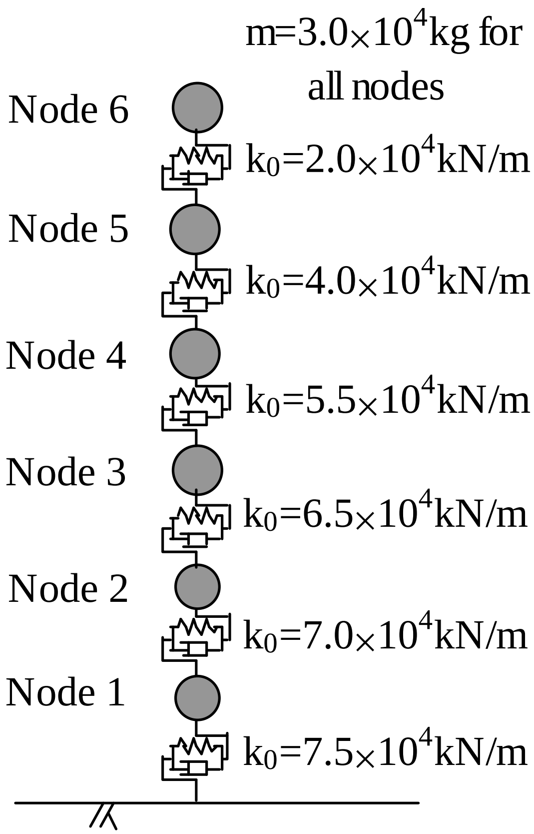


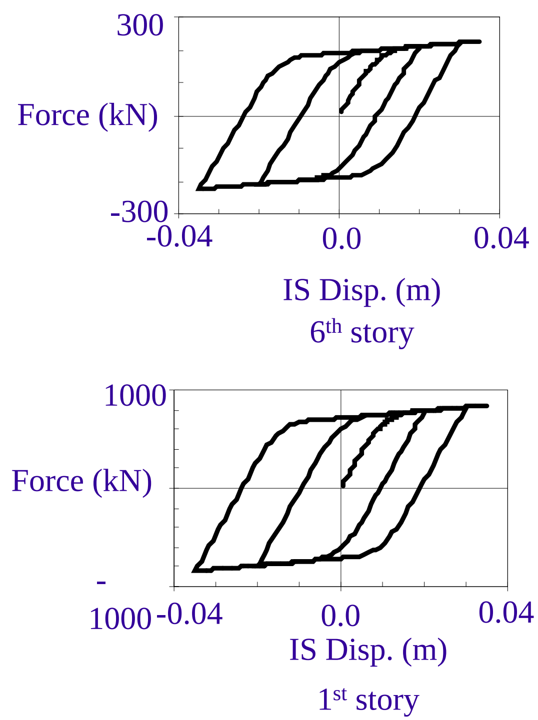



# Applications of TELM



- TELM is easily extended to *MDoF systems* – number of random variables remain the same.




Smooth bilinear hysteresis model

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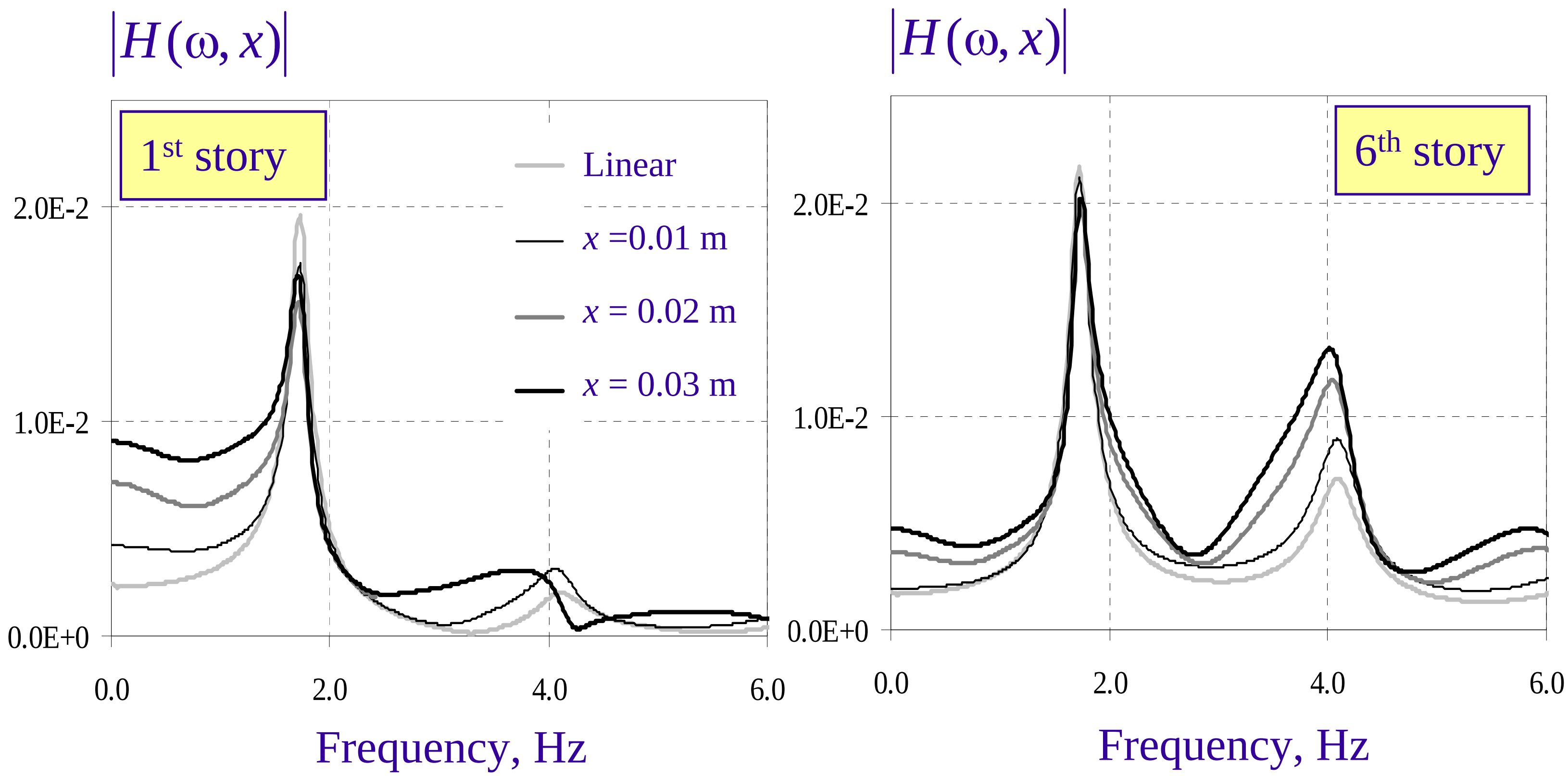
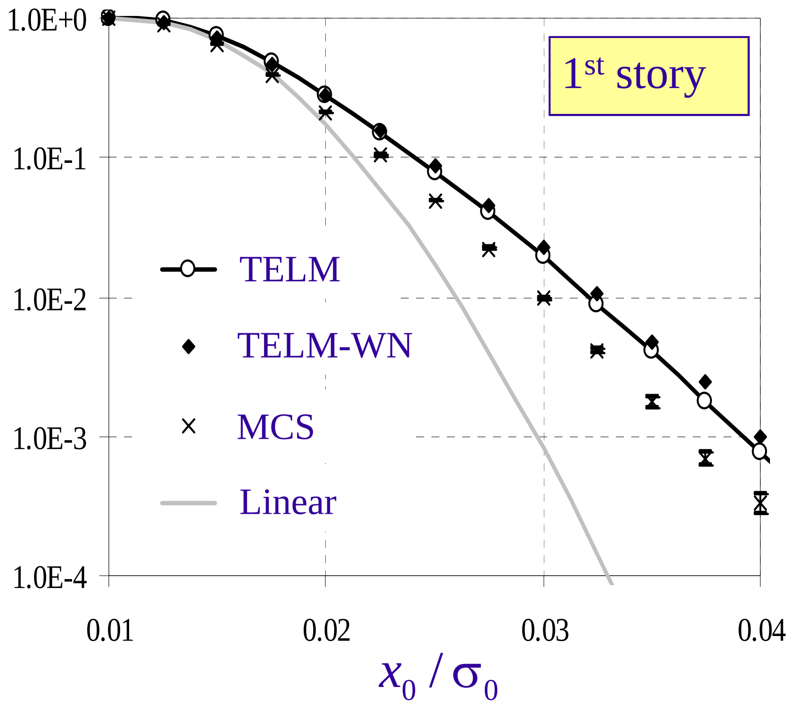
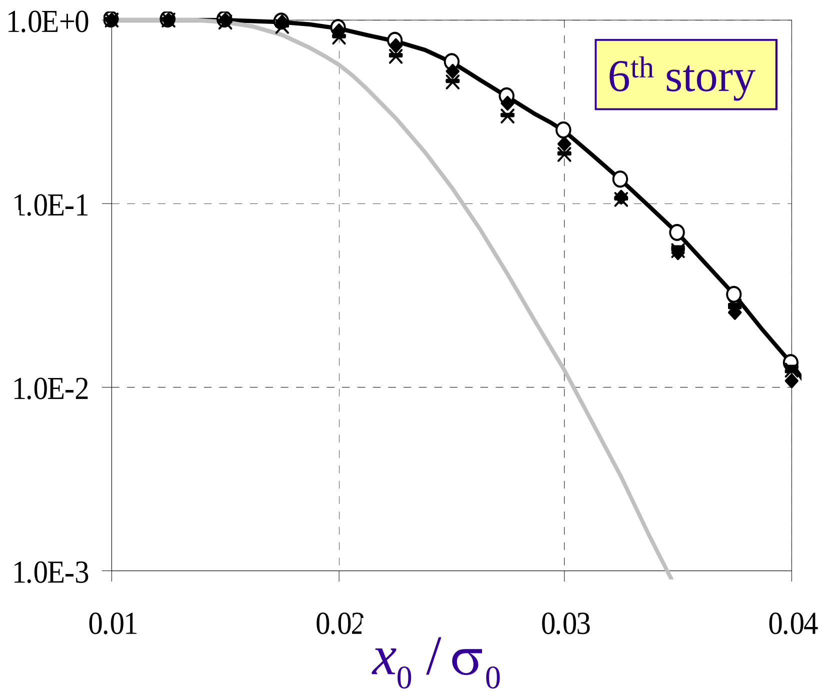


# Applications of TELM



- TELM is easily extended to *MDoF systems* – number of random variables remain the same.




First-passage probability (10s duration)

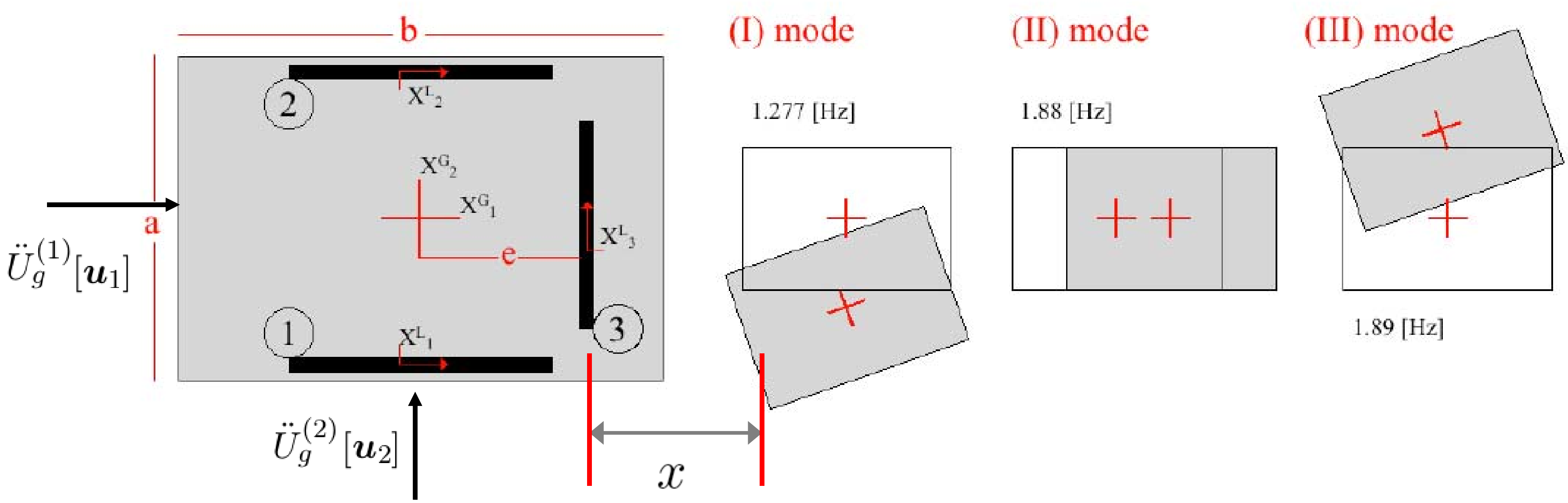
# Applications of TELM



- Multi-component (SI) excitations

$$\mathbf{F}(t) = \begin{bmatrix} \mathbf{s}^{(1)}(t)\mathbf{u}^{(1)} \\ \vdots \\ \mathbf{s}^{(m)}(t)\mathbf{u}^{(m)} \end{bmatrix}$$

a *separate TELS* is identified for each input component.



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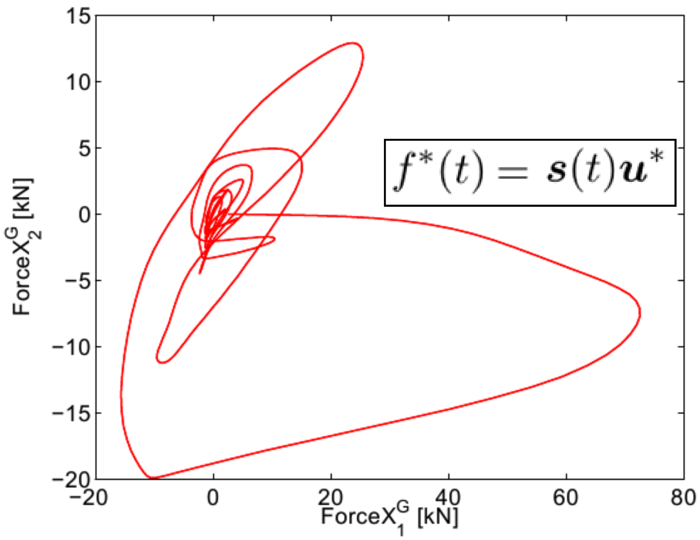
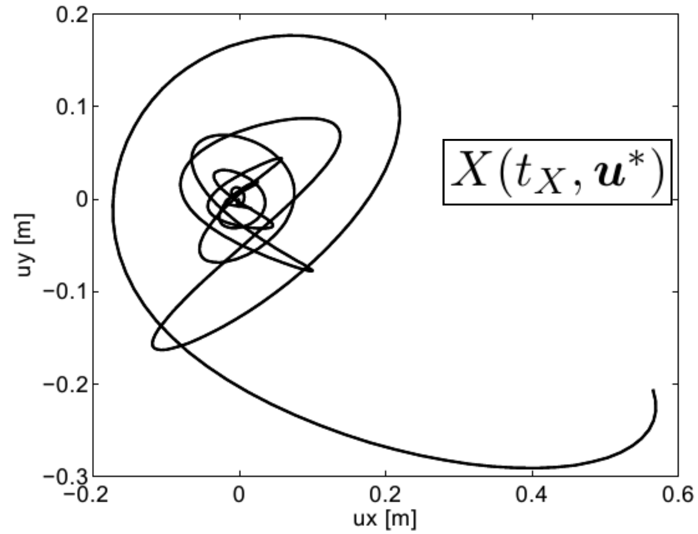
Design point excitation for $x = 3\sigma_0$




Design point response for $x = 3\sigma_0$



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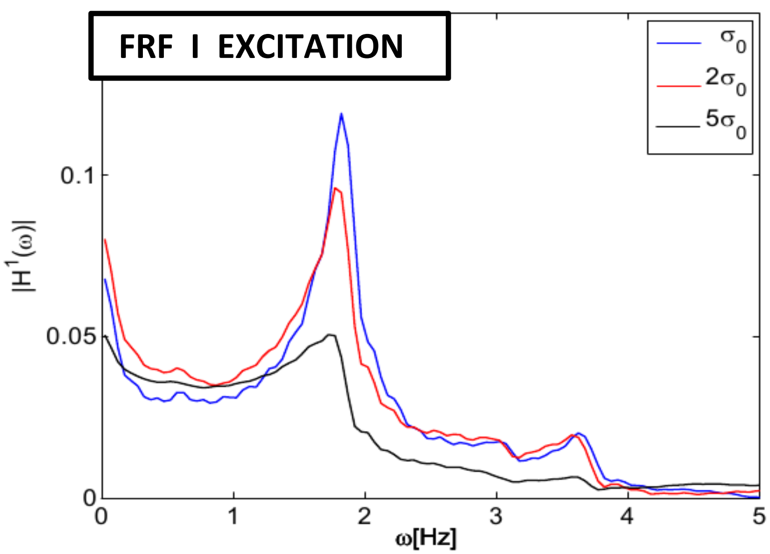
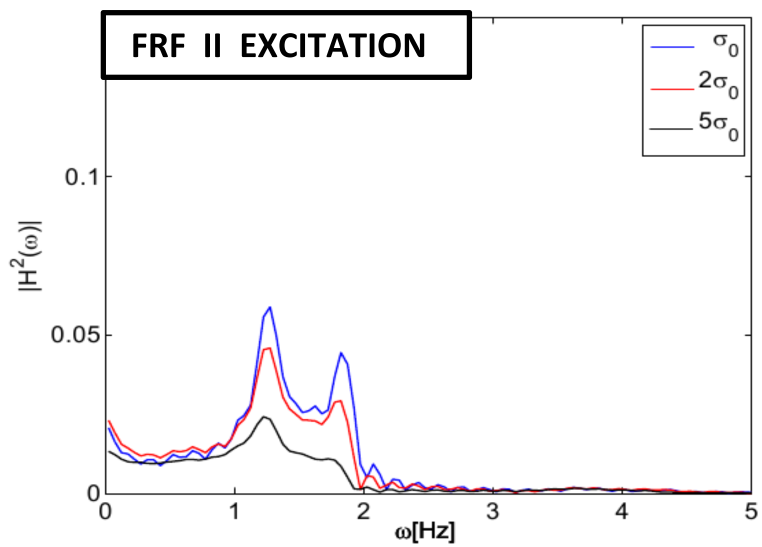
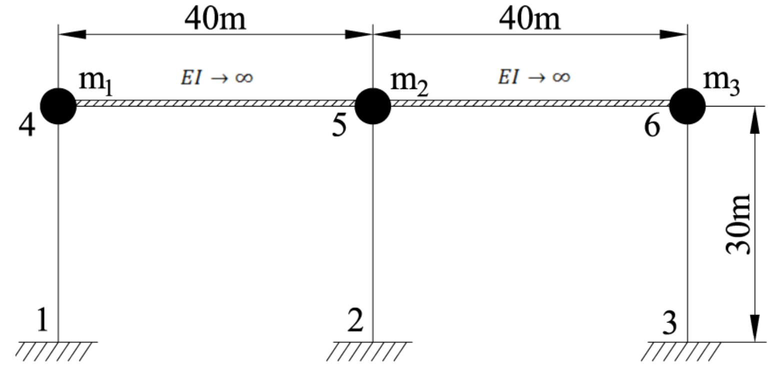


# Applications of TELM



- Multiply-supported inelastic system subject to spatially varying ground motions:




$$D_k(t) = \sum_{p=1}^{n/2} [A_{pk} \cos(\omega_p t) + B_{pk} \sin(\omega_p t)],\ k = 1, \dots, m$$

$m =$ number of support DOFs

$A_{pk}, B_{pk} =$ Fourier coefficients, zero-mean Gaussian random variables, independent for different frequencies, correlated for same frequency different supports

# Applications of TELM



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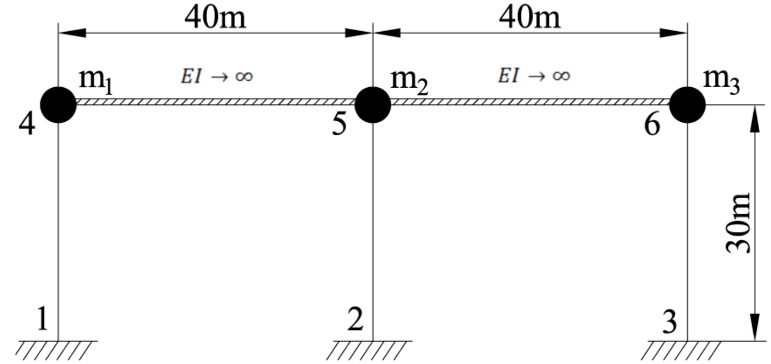


Cases considered:

1. Uniform ground motions
2. Totally incoherent ground motions
3. Wave-passage and partial incoherence
4. Case 3 with variable soil conditions

# Applications of TELM



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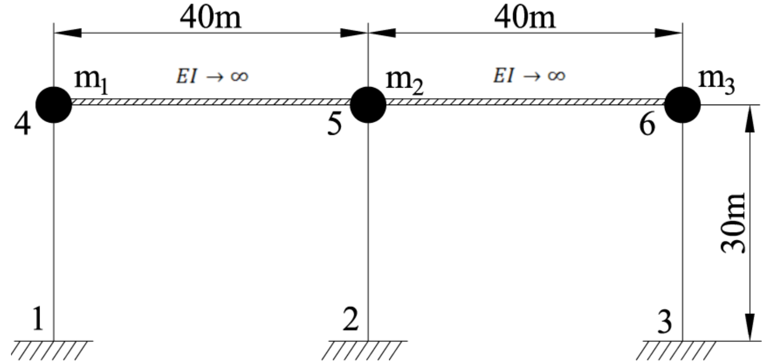
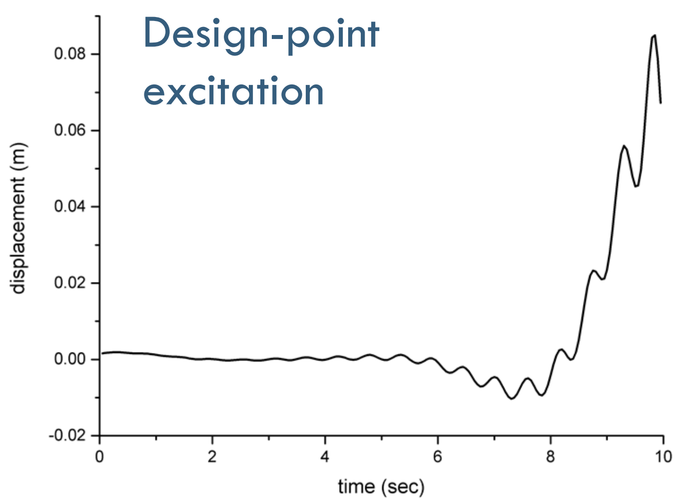
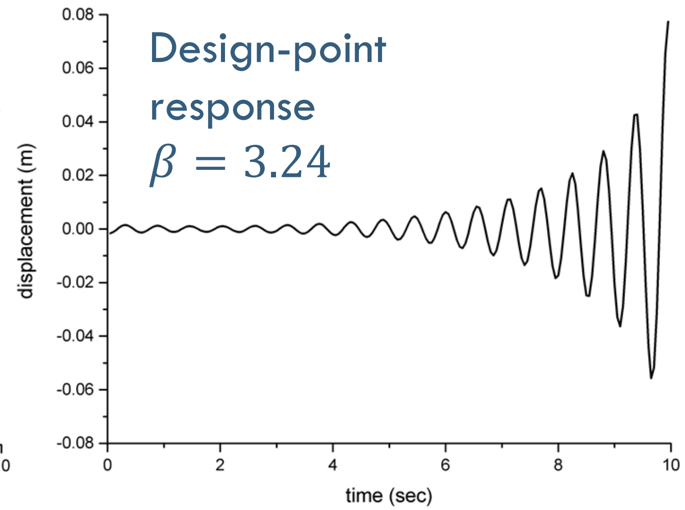



Uniform ground motions



# Applications of TELM



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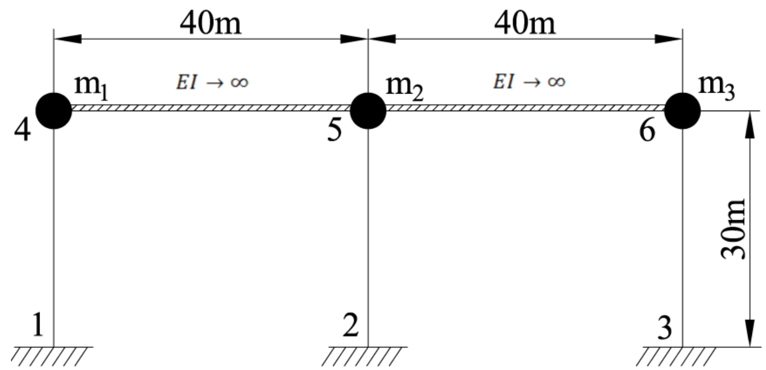
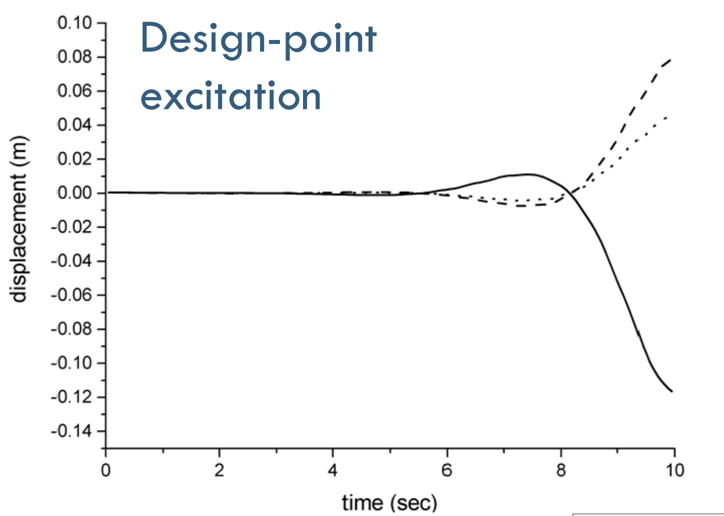
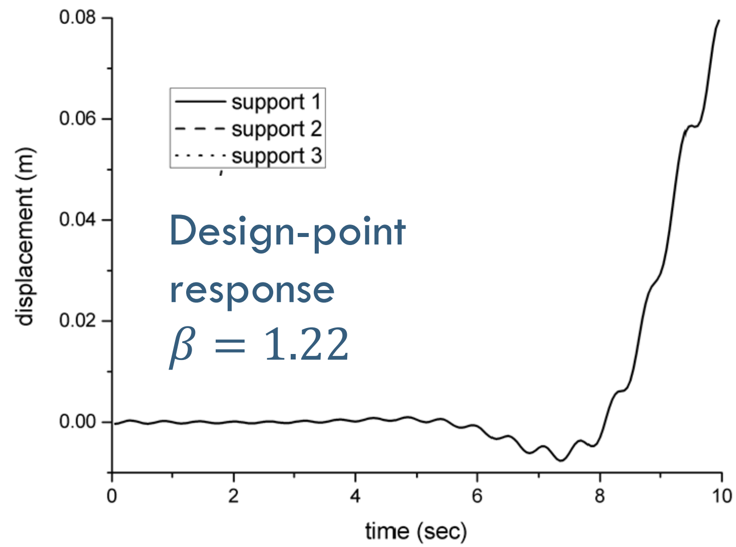



Totally incoherent ground motions



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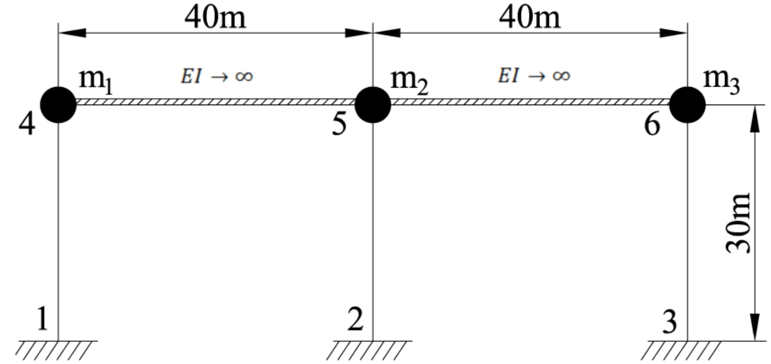
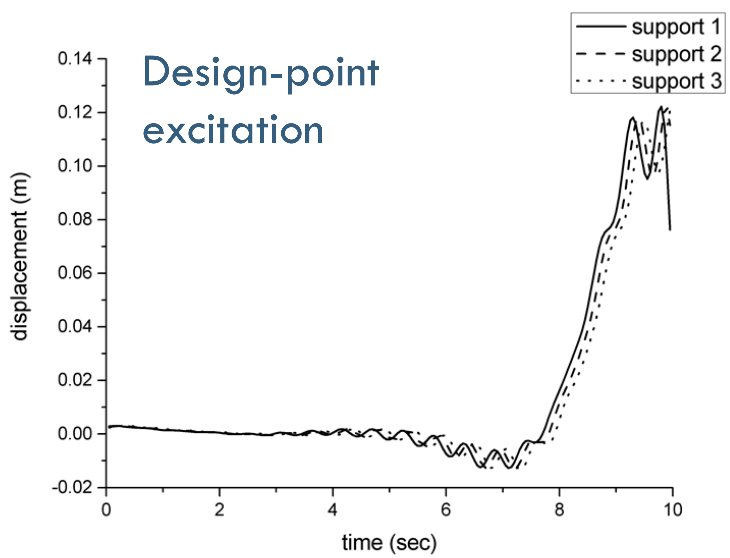
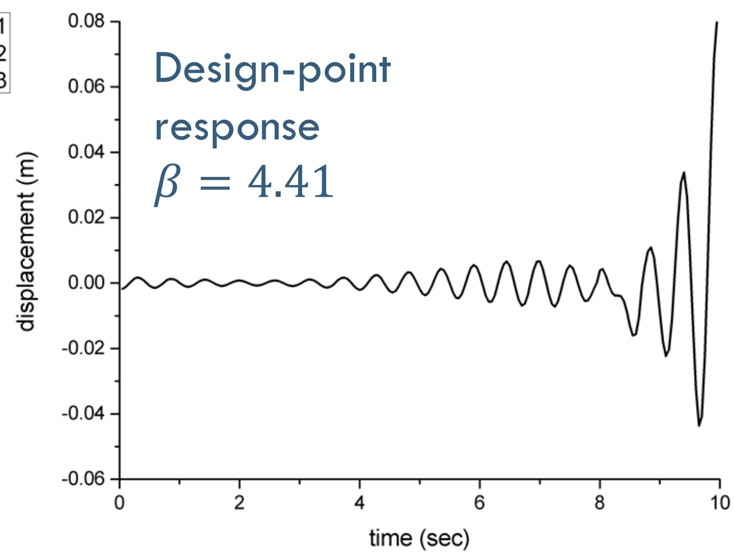



Wave passage and partial incoherence



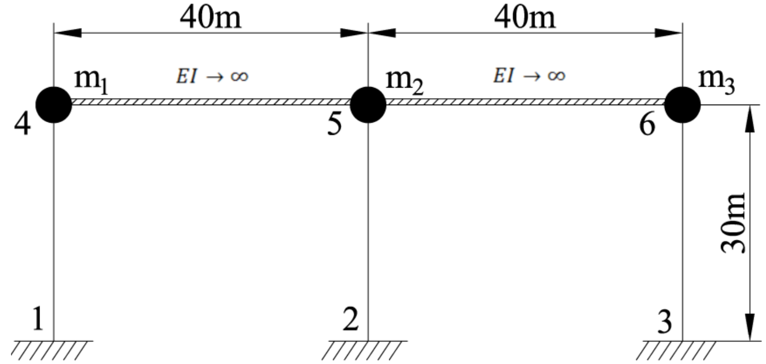



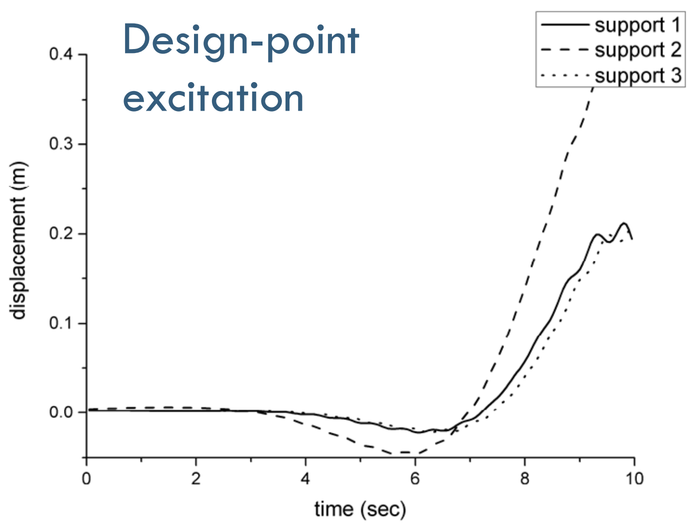
# Applications of TELM

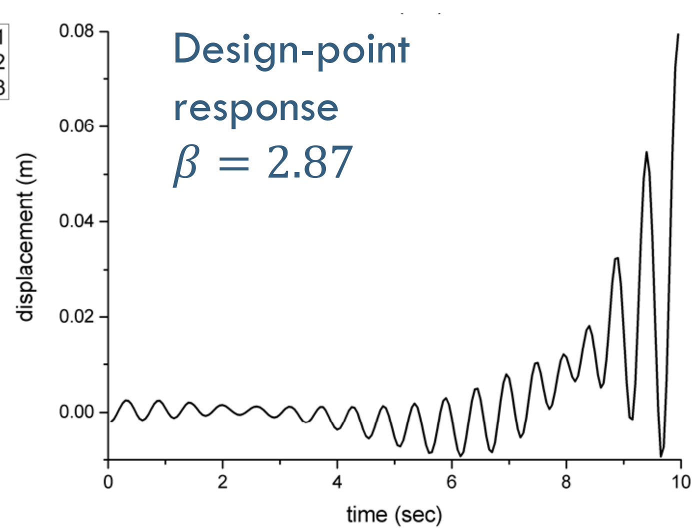


- Multiply-supported inelastic system subject to spatially varying ground motions:

Wave passage and partial incoherence with variable soil



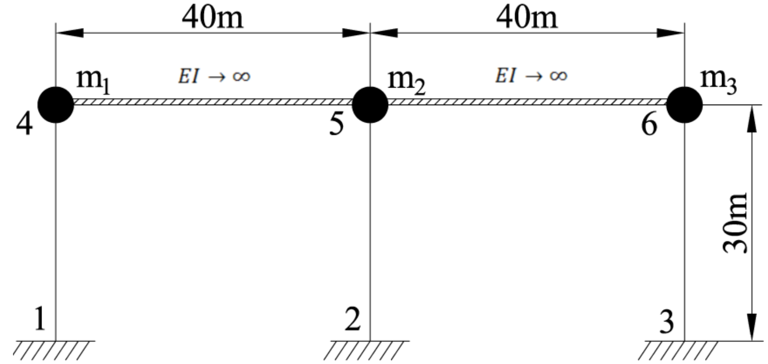



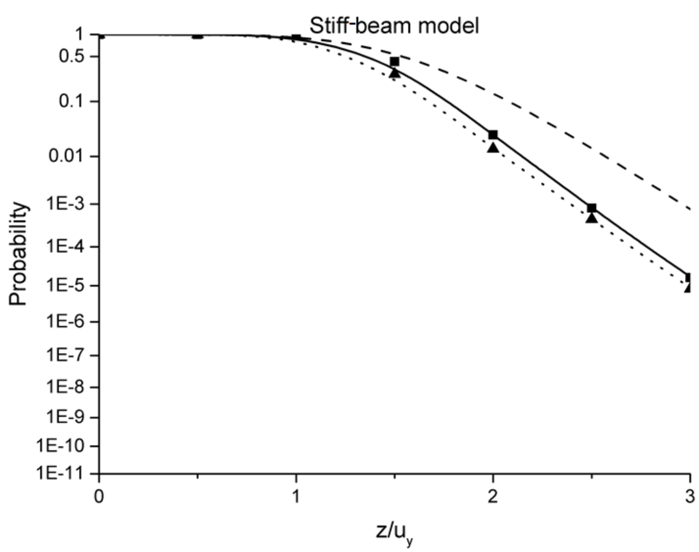
# Applications of TELM

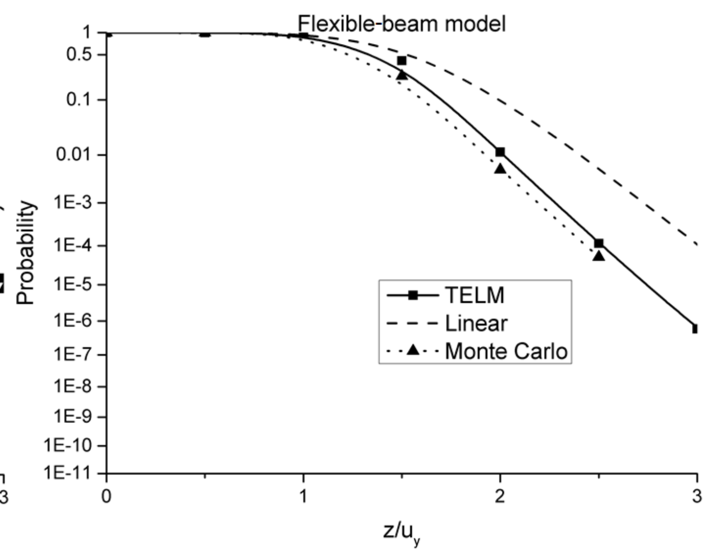


- Multiply-supported inelastic system subject to spatially varying ground motions:




First-passage probability for Case 3



# Challenges and limitations of TELM



- TELM requires repeated computations of $X(t, \mathbf{u})$ and $\nabla_{\mathbf{u}} X(t, \mathbf{u})$ for selected values of $\mathbf{u}$ (typically around 10 times) to find the design point $\mathbf{u}^*$. We use the Direct Differentiation Method (DDM) for this purpose.
- The nonlinear response must be continuously differentiable – must use smooth or smoothened constitutive laws.
- The limit-state surface must be well behaving. TELM does not work well for strongly stiffening systems (e.g., Duffing oscillator with a strong cubic term) or when nonlinearity involves abrupt changes in the system behavior.
- As of now, TELM is not applicable to degrading systems.

# Concluding remarks



- TELM is an alternative equivalent linearization method for nonlinear stochastic dynamic analysis.
- TELM: Is a non-parametric method
  Captures non-Gaussian distribution of nonlinear response
  Offers superior accuracy for tail probabilities
  Is particularly convenient for fragility analysis
  Can be applied to stationary or non-stationary response
  Can be applied to MDoF systems, multi-component excitations, variable support motions
- TELM requires continuous differentiability of the nonlinear response.
- As with other linearization methods, the accuracy of TELM depends on the nature of the nonlinearity.

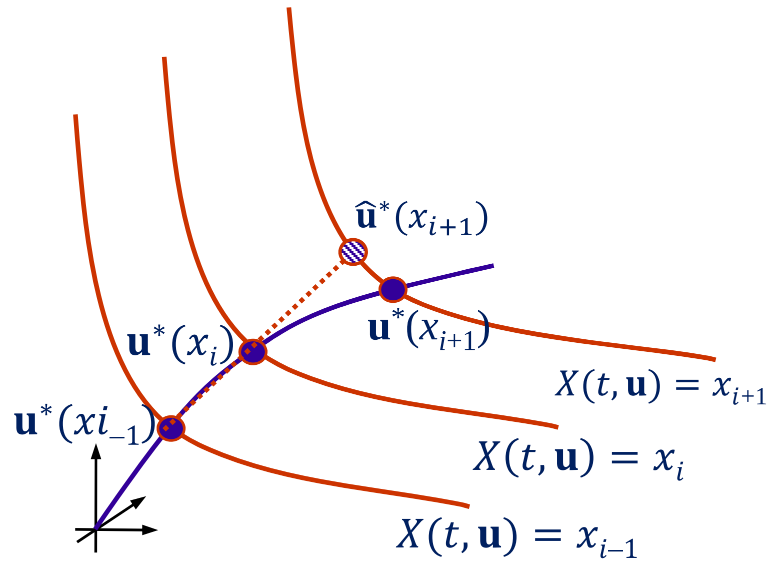
# Thank you!

# Determination of the design point



- Iterative algorithms for solving
  $\mathbf{u}^*(x) = \arg\min\{\|\mathbf{u}\| \mid G(\mathbf{u}, x) = 0\}$
  require repeated computations of $X(t, \mathbf{u})$ and $\nabla_{\mathbf{u}} X(t, \mathbf{u})$.

- For an ordered sequence of thresholds $x_1 < x_2 < \cdots < x_n$, use extrapolated starting points:
  $\hat{\mathbf{u}}^*(x_{i+1}) = \mathbf{u}^*(x_i) + \lambda \frac{\mathbf{u}^*(x_i) - \mathbf{u}^*(x_{i-1})}{\|\mathbf{u}^*(x_i) - \mathbf{u}^*(x_{i-1})\|}$

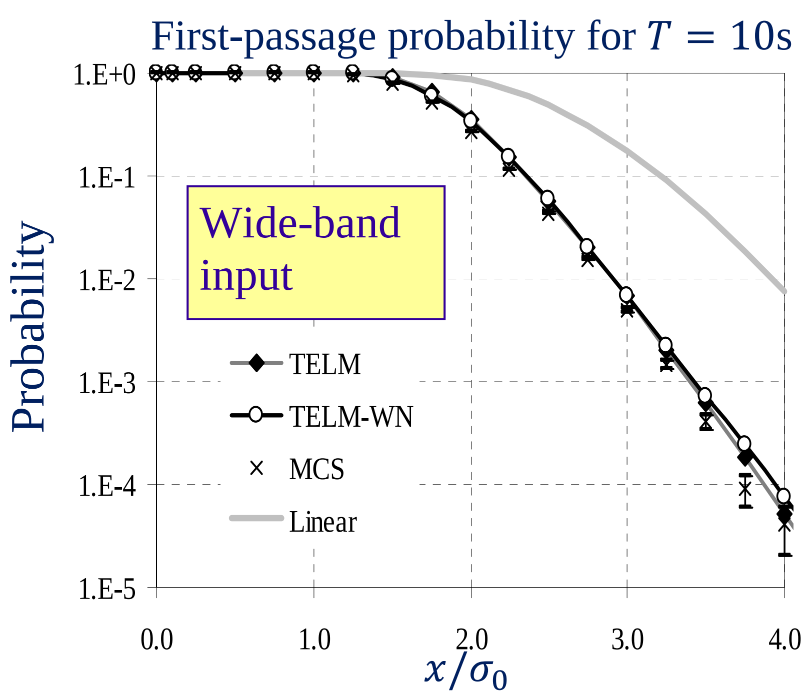


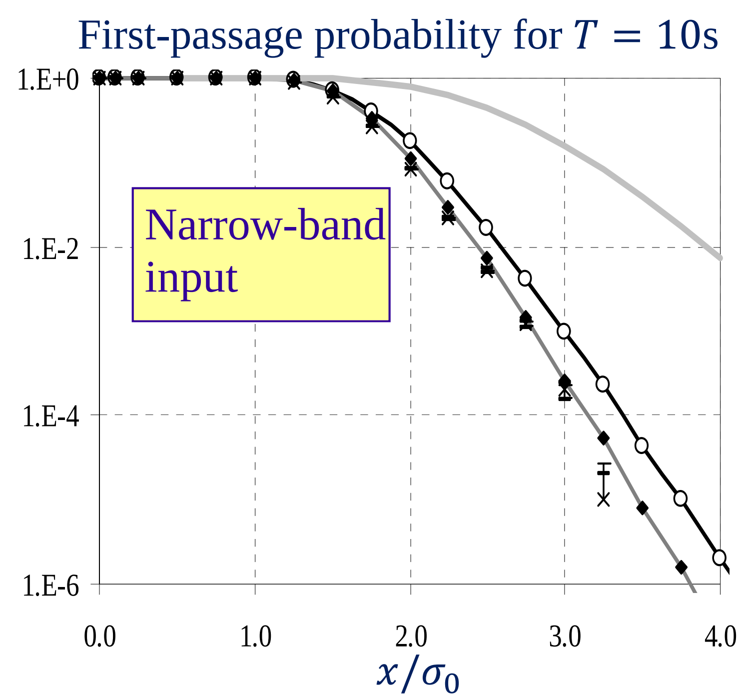
# Characteristics of TELM and TELS



- For broad-band excitations, the TELS is insensitive to the frequency content.

  ⇨ TELS for response to white noise can be used as an approximation for response to *non-white excitations*

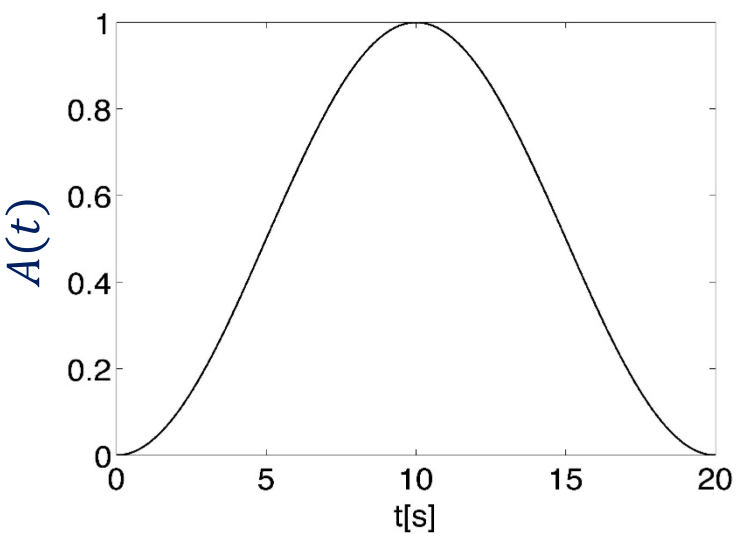


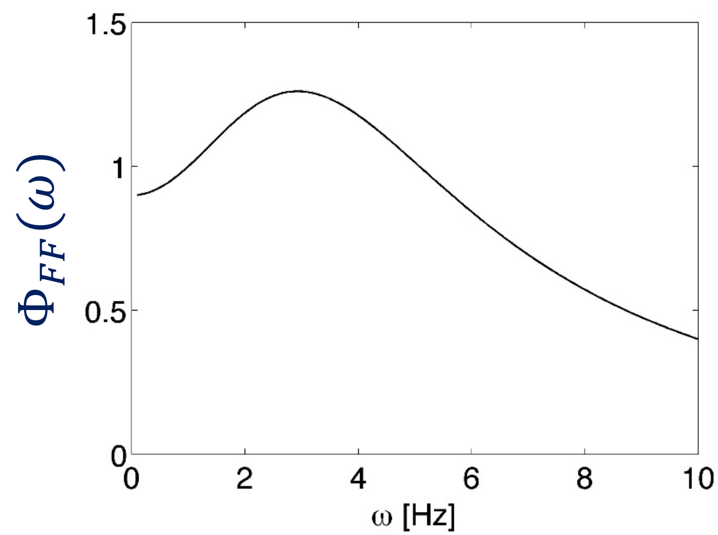



# Characteristics of TELM and TELS



- For non-stationary processes with smooth evolution in time, TELS at a critical time can be used as an approximation for all times.
- Example – response to uniformly modulated process



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